Cash Dividends and Futures Prices on Discontinuous Filtrations

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Abstract

We derive a general formula for the futures price process without the restriction that the assets used in the future margin account are continuous and of finite variation. To do so, we model tradeable securities with dividends which are not necessarily cash dividends at fixed times or continuously paid dividends. A future contract can then be modelled as an asset which pays dividends but has zero value in itself. We show that the futures price is not necessarily a martingale under the equivalent martingale measure, but that it remains a martingale under a new measure which is closely connected to multiplicative Doob-Meyer decompositions. Our definition of self-financing replication is different from some earlier ones, even for assets that do not pay dividends, and we argue that for discontinuous asset price processes it could be more natural than the usual formulation.

Keywords: Financial modeling, Futures, Cash Dividends.

1 Introduction

A futures contract is an exchange-traded standardized contract which gives the holder the obligation to buy or sell a certain commodity (or another financial contract) at a certain date in the future, the delivery date, for a price specified on that day, the settlement price. It should be contrasted with a forward contract, which gives the holder the obligation to buy or sell at a date in the future for a price specified today but paid or received at the future date. The price which is specified today is today’s forward price for the commodity or underlying contract. Forwards are conceptually easier but more complicated in practice, since it assumes that a buyer and a seller agree on cash being paid today and delivery taking place at a future date. Futures have been introduced to standardize the procedure and to make it possible for many people to use prices agreed today for commodities delivered later, without all these people actually having to meet each other to make the exchanges.

Future contracts are bought or sold at zero costs. Today’s futures price for that delivery date tells you for what price you will obtain the commodity at that time, but instead of paying that amount right now (which you would do if you had taken out a forward contract) you pay nothing now. Instead, you open a bank account on the exchange, the so-called margin account. From now until the delivery date\textsuperscript{1} you will receive every day, after the new futures price for your commodity and your delivery date has been specified, the difference between the new futures price and the previous day’s futures price, if this difference is positive. When this difference is negative, the

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corresponding amount is taken from your account. The net effect of this is that you end up paying the futures price at which you obtained your contract in the market.

Since the only difference between the payment stream for a forward and a future is the timing of the payments, it is not surprising that under deterministic interest rates, the forward price and the future price coincide at every point in time if the margin account pays interest. A simple formula exists for the future price if the market of tradeable assets is modeled on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) and the market is complete, i.e. there exists a unique martingale measure \(Q \sim \mathbb{P}\). Under the assumption that the margin bank account process is continuous and of finite variation, the futures price \(m_t\) at time \(t\) for delivery of an asset \(S\) at time \(T\) equals \(m_t = E^Q[S_T|\mathcal{F}_t]\).

Sometimes this is simply taken as the definition of the future price process. However, when the margin bank account is not continuous or not of finite variation, we will argue that this formula no longer holds. This has some relevance for financial practice since some future exchanges allow their clients to use coupon bonds as the investment vehicle for the margin accounts.

In this paper we will analyze future contracts in a general setting. To do so, we will first give a definition of self-financing replication in the presence of cash dividends, i.e. dividends which are not paid continuously. The dividend stream process and the ex-dividend stock price process can be freely specified and we then show how tradeable securities (i.e. the stock process which include dividends) can be generated\(^2\). The future price process is then treated as an application of this more general framework.

The structure of the paper is as follows. In the next section we define a general semimartingale model for tradeable securities that may include dividends. In section 3 we then prove a general representation theorem for tradeable securities in arbitrage-free markets. The future price process will then be investigated in section 4. The last section states conclusions.

2 Modelling General Dividend Processes

In this section, which is a summary of the framework developed in Vellekoop and Nieuwenhuis (2006), we introduce a method to include dividend processes in the modeling of financial securities.

We assume given a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) where the filtration \((\mathcal{F}_t)_{t \in [0,T]}\) is assumed to satisfy the usual conditions, but is not necessarily continuous. Here \(T > 0\) denotes a fixed time horizon. We use the notation \(\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}\) and \(\mathbb{R}_{++} = \{x \in \mathbb{R} \mid x > 0\}\).

The adapted càdlàg stochastic process \(S : \Omega \times [0,T] \rightarrow \mathbb{R}_{++}\) describes the price of one unit of stock ex-dividend. We would then like to define on the same probability space an asset process which may pay dividends. Often this takes the form of a discrete (i.e. cash) dividend equal to \(\hat{D}\) on time \(t_d \in [0,T]\) where \(\hat{D} \in \mathcal{F}_{t_d}\) and such that \(S_{t_d} - \hat{D} > 0\) \((P-a.s.)\). In fact, we would even like to consider cases where an asset pays both continuous and discrete dividends or, even more generally, where the cumulative dividend process is just assumed to be a semi-martingale.

Let \(V_t, S_t\) and \(B_t\) be adapted càdlàg ex-dividend price processes for assets \(V, S\) and \(B\) which are strictly positive and let \(D^V_t, D^S_t\) and \(D^B_t\) be the corresponding càdlàg adapted cumulative dividend processes (which are not necessarily positive), such that \(V_t + D^V_t, S_t + D^S_t\) and \(B_t + D^B_t\) are all semi-martingales. We will assume that \(D^B_0 = D^B_T = D^V_0 = D^V_T = 0\) throughout the paper. The asset \(B\) will often represent a bank account in this setup.

We would like to define the notion of replicability i.e. the idea that the price process of a certain asset \(V\) can be mimicked by trading in other assets.

Definition 2.1. We say that an asset \(V\) can be replicated using assets \(S\) and \(B\) iff there exist adapted and predictable processes \(\phi^S\) and \(\phi^B\) such that for all \(t \in [0, T]\)

\[
\begin{align*}
V_{t-} &= \phi^S_{t-} S_{t-} + \phi^B_{t-} B_{t-} \\
\text{d}(V_t + D^V_t) &= \phi^S_t d(S_t + D^S_t) + \phi^B_t d(B_t + D^B_t)
\end{align*}
\]

\(^2\)We note that in a very interesting recent paper by Korn and Rogers (Korn and Rogers (2004)) the same problem is being treated but there the dividends turn out to be proportional to the underlying asset process, an assumption we do not wish to make here.
where the first equation for \( t = 0 \) should be read as \( V_0 = \phi_0^S S_0 + \phi_0^B B_0 \) (i.e. without taking left-hand side limits).

Note that for processes without dividends we find the classical definition of replication back in (1)-(2), but the lefthand side limits in the first equations are an important difference compared to the case without dividends. Indeed we can no longer say that \( V_t = \phi_t^S S_t + \phi_t^B B_t \) as in the usual formulations in the absence of dividends, but instead \( V_t - \Delta V_t = \phi_t^S (S_t - \Delta S_t) + \phi_t^B (B_t - \Delta B_t) \) which is of course a reformulation of (1) since \( X_t = \Delta X_t = X_{t-} \).

If we define \( \phi_t^S = \psi_t^S V_t - S_t \) and \( \phi_t^B = \psi_t^B V_t - B_t \) then

\[
\frac{dV_t}{V_t} + dD_t^V = \psi_t^S \frac{dS_t + dD_t^S}{S_t} + \psi_t^B \frac{dB_t + dD_t^B}{B_t}
\]

for certain predictable adapted processes \( \psi^S \) and \( \psi^B \) such that \( \psi_t^S + \psi_t^B = 1 \). The interpretation is that the rate of return of \( V \) (which equals the difference in value based on changes in both the ex-dividend price and the dividends, divided by the price before any dividends have been paid out) is based on percentages invested in assets \( S \) and \( B \). Working with percentages guarantees in an intuitive manner that we only consider strategies which do not necessitate cash withdrawal or injection, i.e. it is a convenient way to define self-financing strategies. However, our definition above is slightly more general in the sense that it allows the price processes becoming zero for certain times as well.

Throughout the paper we will assume \( D = 0 \) i.e. our bank account does not pay dividends (or coupons), only interest. Note that we have assumed that \( S + D^S \) is a semi-martingale but we have not assumed it to be continuous.

**Theorem 2.1.** Let \( S + D^S \) and \( B \) be semi-martingales satisfying the conditions stated above. Then

1. there exists a unique asset price process \( S^B \) with \( D^S = 0 \) and \( S^B_0 = S_0 \) that can be replicated with \( \phi^S \equiv 1 \).

2. there exists a unique asset price process \( \tilde{S} \) with \( D \tilde{S} = 0 \) and \( \tilde{S}_0 = S_0 \) such that \( \tilde{S} \) can be replicated using \( S \) only, i.e. such that \( \phi^B \equiv 0 \). This asset price process \( \tilde{S} \) can, together with \( B \), replicate \( S^B \).

3. If an asset \( V \) can be replicated using the assets \( S \) and \( B \), then it can be replicated using the assets \( S^B \) and \( B \).

If an asset \( V \) can be replicated using the assets \( S \) and \( B \), then it can be replicated using the assets \( S \) and \( B \).

**Proof of Theorem 2.1.** To prove the first part of the claim we need to find a process \( \tilde{V} = S^B \) that satisfies our definition of replication with \( \phi^S \equiv 1 \). Since we want \( \phi^S \equiv 1 \) we define \( \phi_t^B = (\tilde{V}_t - S_t)/B_t \), so \( \tilde{V} \) should satisfy

\[
\dot{\tilde{V}} = d(S_t + D_t^S) + \frac{\tilde{V}_t - S_t}{B_t} dB_t.
\]

One can easily show (using Doléans-Dade exponentials) that the process

\[
\tilde{V}_t = S_t + D_t^S - B_t \int_0^t \frac{\tilde{V}_s - S_s}{B_s} dB_s
\]

satisfies the requirements, and that it is the only one to do so.

For the second part, we need to find a process \( V \) such that \( D^V \equiv 0 \), with \( \phi_t^B \equiv 0 \). But this last assumption implies that \( \phi_t^S = V_t - S_t \) so we need to prove that there exists a unique process \( V \) such that \( V_t = S_0 + \int_0^t \frac{\dot{V}_s}{B_s} dB_s + D_t^S \). We can again use Doléans-Dade exponentials to prove existence and uniqueness. The asset \( \dot{V} = S^B \) can be replicated using \( \tilde{V} = \tilde{S} \) and \( B \) since

\[
\dot{\tilde{V}}_t = \phi_t^S \tilde{S}_t + \phi_t^B B_t
\]

\[
\dot{dV}_t = \phi_t^S d\tilde{S}_t + \phi_t^B dB_t
\]
if we take \( \phi^S_t = \frac{S_t - \tilde{S}_t}{\tilde{S}_t} \) and \( \phi^B_t = (\tilde{V}_t - S_t)/B_t \).

For the last part we note that if an asset \( V \) can be replicated using \( S \) and \( B \) we may write
\[
V_t = \phi^S_t V_t^- + \phi^B_t B_t^- \tag{4}
\]
\[
d(V_t + D^V_t) = \phi^S_t d(S_t + D^S_t) + \phi^B_t dB_t \tag{5}
\]
but we can rewrite this as
\[
V_t = \frac{\phi^S_t S_t - \tilde{S}_t}{\tilde{S}_t} + \phi^B_t B_t^- \nonumber
\]
\[
d(V_t + D^V_t) = \frac{\phi^S_t S_t - \tilde{S}_t}{\tilde{S}_t} d\tilde{S}_t + \phi^B_t dB_t \nonumber
\]
so taking \( \phi^S_t = \phi^S_t S_t^-/\tilde{S}_t^- \) shows the first statement. The second statement follows analogously.

The interpretation of the result proven in the Theorem is of course that it should be possible to invest our dividend stream in the bank account or in new stocks and by doing so end up with a process which no longer pays any dividends.

Using partial integration for semi-martingales, we can rewrite formula (3):
\[
S^B_t = S_t + B_t \int_0^t \frac{dD^S_t}{B_{u^-}} + B_t [D^S, B^{-1}]_t
\]
The necessity of the last bracket term to compensate for the fact that paid out cashflows and the bank account may have nonzero covariation was already noted in Norberg and Steffensen (2005).

If we assume that \( B \) is continuous and of finite variation then we simply find \( S^B_t = S_t + B_t \int_0^t \frac{dD^S_t}{B_{u^-}} \). In the special case for just one cash dividend \( D^S_t = \tilde{D}1_{[t, \tau]}(t) \), we can reduce this to \( S^B_t = S_t + 1_{[t, \tau]}(t) \frac{\tilde{D}}{S_t} \). We can simplify the expression for the process \( \tilde{S} \) in a similar way. We then find after a lengthy calculation in which many terms cancel that \( \tilde{S}_t = S_t + 1_{[t, \tau]}(t) \frac{\tilde{D}}{S_t} S_t \), as expected.

If we work on a Brownian filtration, then \( S + D^S \) and \( V + D^V \) are continuous processes, so \( \Delta D^S = -\Delta S \) and the times at which \( D^S \) and \( S \) are discontinuous thus have to coincide. In general we would have for adapted processes \( X \) on this filtration that \( X_t^- = X_t - \Delta X_t = X_t + \Delta D^X_t \) and the first replication equation (1) would then boil down to
\[
V_t + \Delta D^V_t = \phi^S_t (S_t + \Delta D^S_t) + \phi^B_t (B_t + \Delta D^B_t)
\]
which is the classical notion of a gains process to model dividend, and has been introduced earlier in the literature, see for example Duffie (2001). This may seem a natural alternative choice for the first equation in our definition of replication, but it will not generalize in a nice way when we use other filtrations than those generated by Brownian Motion, since we will see later that on filtrations which are not left-continuous we may not always have that \( \Delta D^S = -\Delta S \). On such filtrations our definition is therefore different from the one in Duffie (2001) when dividends are present.

3 Tradeable Assets in Arbitrage-Free Markets

We now consider an arbitrage-free market with the assets \((\tilde{S}, B)\) in it. We know that there exists a measure \( Q \), equivalent to our original measure \( P \), such that \( \tilde{S}/B \) is a martingale under \( Q \). We use the common notation \( Z = X_{t-} - Y \) for a process \( Z \) satisfying \( dZ_t = X_t - dY_t \).

**Definition 3.1.** We say that \( V \) is the price process of a tradeable asset iff

4
1. It can be replicated using $\tilde{S}$ and $B$

2. The process $\mathcal{D}(V)$ is a martingale under $\mathbb{Q}$, where

\[
\mathcal{D}(V) = \frac{V + D^V}{B} - D^V \cdot B^{-1}
\]

Due to the corollary proven above, we might as well have required that $V$ can be replicated using $S^B$ and $B$.

We noted before in (6) that we may rewrite $\mathcal{D}(V)_t$ as $\mathcal{D}(V) = \frac{V_t}{B_t} + B^{-1} \cdot D^V + [D^V, B^{-1}]$. The main point of the definition given above is that we would like $\mathcal{D}$ to be a martingale, and not just a local martingale. That it is a local martingale is already guaranteed by the first part of the definition, as the following result shows. This representation theorem is the main result of the paper, which shows how the usual martingale representation theory for assets without dividends carries over to our more general case.

**Theorem 3.1.** If an asset price process $V$ can be replicated using $S$ and $B$ then there exists an adapted predictable process $\phi$ such that

\[
d\mathcal{D}(V)_t = \phi_t \frac{d\tilde{S}_t}{B_t}
\]

**Proof.** This follows directly after lengthy calculations by a straightforward application of Ito’s rule for divisions of semimartingales:

\[
d\frac{X_t}{Y_t} = \frac{dX_t}{Y_t} - \frac{dX_t dY_t}{Y_t^2} + \frac{X_t d[Y, Y]_t}{Y_t^3} + \frac{\Delta Y_t}{Y_t} \left( \frac{X_t - \Delta Y_t}{Y_t} - \frac{\Delta X_t}{Y_t} \right).
\]

We use the fact that if $V$ can be replicated using $S$ and $B$, it can be replicated using $\tilde{S}$ and $B$ by the previous corollary, so there exist $\phi^{\tilde{S}}$ and $\phi^B$ such that

\[
V_t = \phi_t^{\tilde{S}} \tilde{S}_t + \phi_t^B B_t \tag{6}
\]

\[
d(V_t + D^V_t) = \phi_t^{\tilde{S}} d\tilde{S}_t + \phi_t^B dB_t \tag{7}
\]

since $D^{\tilde{S}} = D^B \equiv 0$. Applying the rule above with $X_t = V_t + D^V_t$ and $Y_t = B_t$ then gives the result. \qed

We have thus proven that asset price processes $V$ that can be constructed in a self-financing manner using stock and the bank account, inherit the local martingale property from the underly- ing assets: if the discounted version of $\tilde{S}$ is a local martingale under $\mathbb{Q}$, then so is $\mathcal{D}(V)$, the properly discounted version of $V$ and its dividend process $D^V$. This will allow us to apply the usual theory for option pricing in arbitrage-free markets without dividends.

Note that we allow tradeables here to have dividend processes. Alternatively we could say that $V$ is a tradeable whenever $D^V \equiv 0$ and $\frac{V}{B}$ is a $\mathbb{Q}$-martingale, but we will see in the applications of the next section that this would be too restrictive for many financial applications.

Since $\mathcal{D}(V)_t$ is a $\mathbb{Q}$-martingale we have that $\mathbb{E}^\mathbb{Q}[\Delta \mathcal{D}(V)_t \mid \mathcal{F}_s] = \mathcal{D}(V)_s$ and taking limits $s \uparrow t$ we find that $\mathbb{E}^\mathbb{Q}[\Delta \mathcal{D}(V)_t \mid \mathcal{F}_{t-}] = 0$. So when $B$ is continuous and of finite variation we must have that

\[
\mathbb{E}^\mathbb{Q}[\Delta V_t + \Delta D^V_t \mid \mathcal{F}_{t-}] = 0
\]

This expression immediately shows that on left-continuous filtrations (such as those generated by Brownian Motion) where $\mathcal{F}_{t-} = \mathcal{F}_t$, we must have that $\Delta V = -\Delta D^V$ since both $V$ and $D^V$ are adapted. But if the underlying filtration is not left-continuous this is no longer necessary, even if
cash dividend payments are announced in advance (i.e. when $\Delta D$ is $\mathcal{F}_{t-}$-measurable). We then only know that

$$
\mathbb{E}^Q[\Delta V_t | \mathcal{F}_{t-}] = -\Delta D_t
$$

so the jump in the ex-dividend process of a tradeable does not necessarily cancel the jump due to a dividend payment. This was already noted in Heath and Jarrow (1988) and Battauz (2002). In the last paper an asset price model is formulated in which $D_t = D_{1\geq t}$ with $D$ and $t_D$ deterministic, and $\Delta V_{t_D} = -D + Y(V_{t_D} - D)$ for a stochastic variable $Y$ with support $[-1, 1]$ and such that $\mathbb{E}^Q[Y | \mathcal{F}_{t_D-}] = 0$. This provides a nice example of a tractable dividend model where $\Delta V \neq -\Delta D$.

4 Futures

We will now show how the framework developed so far can be applied to future price processes. As discussed in the introduction, the futures contract has three essential elements:

- Going long or short any number of futures contracts is free at all times
- With every future contract we enter, we can associate a margin account in which the differences between the current and previous futures price is being paid (if we are long one contract) or withdrawn (if we are short one contract).
- This margin account earns interest.

We will use these three elements as the basis of a definition of a futures price.

Definition 4.1. We call $m : \Omega \times [0, T] \to \mathbb{R}$ the futures price process associated with delivery of asset $S$ at time $T$ if the following holds:

- $m$ is a semi-martingale and $m_T = S_T$
- For all bounded previsible processes $\psi$ the following process $M$ is a tradeable:

$$
\begin{align*}
\left\{ 
  \begin{array}{ll}
  dM_t &= M_t - \frac{d\mathcal{M}_t}{\mathcal{F}_{t-}} + \psi_t dm_t \\
  M_0 &= 0 
  \end{array}
\right. 
\end{align*}
$$

Notice that delivery involves the ex-dividend price, and not the price of the tradeable.

We will use the notation $M^\psi$ for the process $M$ to remind ourselves that it depends on the process $\psi$. Note that the process $\psi$ in the definition above has the interpretation of a futures trading strategy: $\psi_t$ represents the number of futures contracts in our position at time $t$. Our definition reflects the fact that we may enter the futures market at any time at zero costs. What we do is to ‘invest’ the proceeds of the futures strategy $\psi$ into the margin-account $M$ which earns the riskfree rate.

This approach is different from the usual one (see for example Bjork (2004)) where margin accounts are never taken into account explicitly. The only exception we know of is the work of Duffie and Stanton (1992) in which the margin account is mentioned directly. Our treatment here is inspired by the paper by Pozdnyakov and Steele on the martingale framework for futures pricing, Pozdnyakov and Steele (2004), but our definition differs from theirs. We only impose that $m$ is such that $M^\psi/B$ is a $\mathbb{Q}$-martingale on $[0, T]$ (i.e. that $M^\psi$ is a tradeable in economic parlance). Another difference with their approach is that we introduce a whole collection of tradeables from the very beginning and this is completely in line with the fact that one may enter a futures contract at any time in real life.

The following results is then immediate:

Lemma 4.1. The margin account process can be replicated using a zero ex-dividend process with pays continuous dividends equal to the futures price.
Proof. Taking \( \phi^S_t = \psi_t, \phi^B_t = M_t/B_t \) and \( \hat{S}_t = 0, D^S_t = m_t \) replicates \( V_t = M_t \) with \( D^V_t = 0 \), see equations (1)-(2).

Let \( \mathcal{E}(X) \) denote the Doléans-Dade exponential of a process \( X \) with initial value \( \mathcal{E}(X)_0 = 1 \).

We are now able to formulate our main Theorem.

**Theorem 4.1.** Assume that the jumps of \( 1/B \) are bounded, so the semimartingale \( 1/B \) is special and has a unique Doob-Meyer decomposition \( 1/B = M^{1/B} + A^{1/B} \) into a \( \mathbb{Q} \)-martingale \( M^{1/B} \) and a predictable finite variation process \( A^{1/B} \). Define \( H = \mathcal{E}(\hat{A} \cdot M^{1/B}) \) with
\[
\hat{A} = \frac{B_-}{1 + B_- \Delta A^{1/B}}
\]
and assume that \( \mathbb{E}^Q H_T = 1 \). Then the futures price process is given by
\[
m_t = \mathbb{E}^Q [S_T \mid \mathcal{F}_t]
\]
where the measure \( \mathbb{H} \) is defined by \( d\mathbb{H}/d\mathbb{Q} = H_T \).

Proof. Let \( M \) be a margin account process for \( \psi \equiv 1 \) with \( m \) as given above. We have
\[
dm_t = d(\mathcal{M}^H_t) - \mathcal{M}^H_{t-} dB_t = B_t d(\mathcal{M}^H_t) + dB_t,
\]
so \( \frac{1}{m} \cdot m = \hat{M} + [\hat{M}, \hat{B}] \) with \( \hat{M} = \frac{B_-}{m_-} \cdot \mathcal{M}^H \) a \( \mathbb{Q} \)-martingale and with \( \hat{B} = \frac{1}{m_-} \cdot B \).

Since \( m_T = S_T \), it is clear that we are done if we prove that \( m \) is an \( \mathbb{H} \)-martingale or, equivalently, if we prove that \( mH \) is a \( \mathbb{Q} \)-martingale. We denote \( G = \frac{1}{m} \cdot H \) and consider

\[
\begin{align*}
\frac{d(mH)}{m_-H_{u-}} &= \frac{dm_t}{m_-} + \frac{dH}{H_{u-}} + \frac{d[mH]}{m_-H_{u-}} = d\hat{M}_u + [\hat{M}, \hat{B}]_u + dG_u + d[\hat{M}, G]_u + d[\hat{M}, \hat{B}, G]_u, \\
&= d\hat{M}_u + dG_u + d[\hat{M}, \hat{B} + G + [\hat{B}, G]]_u,
\end{align*}
\]

where we have used the fact that \( d[\hat{M}, \hat{B}, G]_u = \Delta \hat{M}_u \Delta \hat{B}_u \Delta G_u = d[\hat{M}, [\hat{B}, G]]_u \). The first two terms in (9) correspond to \( \mathbb{Q} \)-martingales \( \hat{M} \) and \( G \) so we are done if we show that
\[
[\hat{M}, \hat{B} + G + [\hat{B}, G]] = \mathcal{P} \cdot [\hat{M}, H^B]
\]
is a \( \mathbb{Q} \)-martingale. We will do so by proving that \( \mathcal{P} \cdot (HB) \) is a finite variation process and predictable. We first calculate
\[
\mathcal{P} \cdot H = \hat{A} \cdot M^{1/B} = B_- \cdot M^{1/B} + (\hat{A} - B_-) \cdot M^{1/B} \]
\[
= B_- \cdot M^{1/B} - \frac{(B_-) \Delta A^{1/B}}{1 + B_- \Delta A^{1/B}} \cdot M^{1/B} = B_- \cdot M^{1/B} - \frac{(B_-) \Delta A^{1/B}}{1 + B_- \Delta A^{1/B}} \Delta M^{1/B}
\]
\[
\mathcal{P} \cdot B = -B_- \cdot \mathcal{P} + F^B
\]

where \( F^B \) is a process of finite variation, so
\[
\mathcal{P} \cdot (HB) = B \cdot H + \mathcal{P} \cdot B + \mathcal{P} \cdot [H, B]
\]
\[
= -B_- \cdot A^{1/B} - \frac{(B_-) \Delta A^{1/B}}{1 + B_- \Delta A^{1/B}} \Delta M^{1/B} + F^B + \mathcal{P} \cdot [H, B],
\]

has finite variation, since \( A^{1/B} \), pure jump processes and bracket processes all have finite variation. But \( HB \) is also predictable, since
\[
HB = BH_- (1 + \frac{\Delta H}{H_{u-}}) = BH_- (1 + \hat{A} \Delta M^{1/B}) = BH_- (1 + \frac{B_- \Delta M^{1/B}}{1 + B_- \Delta A^{1/B}})
\]
\[
= BH_- \frac{1 + B_- \Delta A^{1/B}}{1 + B_- \Delta A^{1/B}} = BH \frac{B_- \Delta A^{1/B}}{1 + B_- \Delta A^{1/B}} = \mathcal{P} \cdot [H, B].
\]

But when \( HB \) is of finite variation and predictable, \( \mathcal{P} \cdot (HB) \) is predictable, which concludes the proof.

\[ \square \]
Remarks.

The theorem shows that the existence and uniqueness of the futures price process $m$ is guaranteed if the jumps of $1/B$ are bounded, and the local martingale $\mathcal{E}(\bar{A} \cdot M^{1/B})$ is actually a martingale.

If $B$, and therefore $\bar{B}$, is continuous and of finite variation then we can take $G = 0$ and $H = \mathbb{Q}$, since in that case $[\bar{M}, \bar{B}]$ is already a $\mathbb{Q}$-martingale. This leads to the property that $m$ is already a martingale under $\mathbb{Q}$ itself, which is often taken as the definition of a future price process. If $B$ is continuous but not necessarily of finite variation then the formula for the Radon-Nikodym derivative $\frac{d\mathbb{H}}{d\mathbb{Q}}$ can be simplified, since in this case $\mathcal{E}(\bar{A} \cdot M^{1/B}) = \mathcal{E}(-M^B)$.

Note that the change of measure we use here is closely related to the multiplicative Doob-Meyer decomposition, discussed in a recent paper Jamshidian (2007), and also in a different form in Jacod and Shiryaev (2002). Indeed, we see that in the proof the essential elements are that (1) $H$ is a martingale, and (2) $HB$, and therefore $1/KB = (1/B)/H$, is a predictable process of finite variation under $\mathbb{Q}$, which is achieved by a multiplicative Doob-Meyer decomposition for $1/B$.

5 Conclusions

We have shown how dividends can be modeled consistently in arbitrage-free markets by the introduction of tradeable securities without dividends that can be replicated using underlying assets with dividends. Our definition of what replication should mean in the presence of dividends differs from some other definitions when filtrations are not continuous. We believe it provides a natural concept for the modelling of dividends, as witnessed for example by the future price processes given in the previous section.

It is obvious that when dividends are present, the ex-dividend process cannot be a martingale under the usual equivalent martingale measure after discounting. We have shown in particular that the futures price process can only remain a martingale under a different equivalent measure $\mathbb{H}$. The precise consequences for this result for the use of futures to hedge more complex derivatives will be the subject of further research.

References


