ESTIMATIONS OF SOLUTIONS CONVERGENCE OF HYBRID SYSTEMS WITH DELAY

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Introduction

Let consider the hybrid dynamic system with switches consist of a set of subsystems which are linear differential-difference equations

\[ \dot{x}(t) = A_i x(t) + B_i x(t - \tau), \quad i = 1, \ldots, n, \quad x(t) \in \mathbb{R}^n, \quad t \geq 0. \] (1)

Each of subsystems describes dynamic on a fixed finite time interval \( t_{i-1} \leq t < t_i, \quad i = 1, \ldots, N, \quad t_0 = 0. \) Subsystems can be stable or unstable. It is required to estimate the size of deviation of solutions \( x(t) \) of logic-dynamical system (1) from origin at the final moment \( t = t_N \) if it is known the initial indignation.

Here and further the following vector and matrix norms are used

\[ |A| = \{ \lambda_{\max}(A^T A) \}^{1/2} \] (2)

\[ |x(t)| = \left\{ \sum_{i=1}^{n} x_i^2(t) \right\}^{1/2} \]

\[ \|x(t)\|_\tau = \max_{-\tau \leq s \leq 0} \{|x(s + t)|\} \]

\[ \|x(t)\|_{\tau,\beta} = \left\{ \int_{-\tau}^{0} e^{\beta s} |x(t + s)|^2 ds \right\}^{1/2} \]

\( \lambda_{\max}(\cdot), \quad \lambda_{\min}(\cdot) \) — the greatest and least eigenvalues of the corresponding symmetric, positively certain matrixes.

In this paper we use Lyapunov-Krasovsky functional in square form. The multiplier of exponential view at the integral allows to get estimation for derivative with an element containing an integral.
Preliminary Result

In this section we obtain the estimation of convergence of solution \( x(t) \) of a linear stationary subsystem with delay

\[
\dot{x}(t) = Ax(t) + Bx(t - \tau),
\]

(3)
determined on an interval \( t_0 \leq t \leq t_1 \). For obtaining of estimation of solutions we use a functional in such view

\[
V [x(t), t] = e^{\gamma t} \left\{ x^T(t)Hx(t) + \int_{-\tau}^{0} e^{\beta s} x^T(t + s)Gx(t + s) ds \right\}.
\]

(4)

Let us denote

\[
\varphi_{11}(H) = \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)},
\]

\[
\varphi_{12}(G, H) = \frac{\lambda_{\max}(G)}{\lambda_{\min}(H)},
\]

\[
\varphi_{21}(G, H) = \frac{\lambda_{\max}(H)}{\lambda_{\min}(G)},
\]

\[
\varphi_{22}(G) = \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)},
\]

\[
S[G, H] = \begin{bmatrix}
-A^T H - HA - G & -HB \\
-B^T H & G
\end{bmatrix}.
\]

(5)

We obtain the following result.

**Theorem 1.** Let there exist positively certain matrixes \( G \) and \( H \) at which the matrix \( S[G, H] \) also is positively determined. Then the system (3) is asymptotic stable and for its solutions \( x(t) \) are fair following top exponential estimations of convergence

\[
|x(t)| \leq \left[ \sqrt{\varphi_{11}(H)} |x(0)| + \sqrt{\varphi_{12}(G, H)} \|x(0)\|_{\tau, \beta} \right] \exp \left\{ -\frac{1}{2} \varsigma t \right\},
\]

(6)

\[
t \geq 0,
\]

and

\[
\|x(t)\|_{\tau, \beta} \leq \left[ \sqrt{\varphi_{21}(G, H)} |x(0)| + \sqrt{\varphi_{22}(G)} \|x(0)\|_{\tau, \beta} \right] \exp \left\{ -\frac{1}{2} \varsigma t \right\},
\]

(7)

\[
t \geq 0.
\]
\[ \varsigma (\beta, \gamma) = \min \left\{ \frac{\lambda_{\min} (S [G, H])}{\lambda_{\max} (H)}, \beta \frac{\lambda_{\min} (G)}{\lambda_{\max} (G)} + \gamma \left[ 1 - \frac{\lambda_{\min} (G)}{\lambda_{\max} (G)} \right] \right\}, \quad (8) \]

The value \( \beta \geq 0 \) can be arbitrary for

\[ \lambda_{\min} (S [G, H]) \geq \lambda_{\max} (G). \]

And

\[ \beta \leq \frac{1}{\tau} \ln \left\{ \frac{\lambda_{\max} (G)}{\lambda_{\max} (G) - \lambda_{\min} (S [G, H])} \right\}. \]

if

\[ \lambda_{\min} (S [G, H]) < \lambda_{\max} (H). \]

The value \( \gamma \) satisfies to a condition \( \gamma \leq \beta \).

Proof. For the proof we’ll use Lyapunov-Krasovsky functional of kind (4) with positively certain matrixes \( G \) and \( H \). It satisfies to the following bilateral estimations

\[ e^{\gamma t} \left\{ \lambda_{\min} (H) \| x(t) \|^2 + \lambda_{\min} (G) \| x(t) \|^2 \right\} \leq V [x(t), t] \leq e^{\gamma t} \left\{ \lambda_{\max} (H) \| x(t) \|^2 + \lambda_{\max} (G) \| x(t) \|^2 \right\} \]

We’ll find an estimation for its derivative in force of system (3). We’ll make replacement \( t + s = \xi \). Then functional will transform to

\[ V [x(t), t] = e^{\gamma t} \left\{ x^T (t) H x(t) + \int_{t-\tau}^{t} e^{\beta (t-\xi)} x^T (\xi) G x(\xi) d\xi \right\}. \quad (10) \]

We’ll calculate a full derivative of transformed functional (10) along solutions \( x(t) \) of system (3). We obtain

\[ \frac{d}{dt} V [x(t), t] = -e^{\gamma t} \left\{ (\beta - \gamma) \int_{t-\tau}^{t} e^{\beta (t-\xi)} x^T (\xi) G x(\xi) d\xi \right\} - \]

\[ -e^{\gamma t} \left( x^T (t), x^T (t - \tau) \right) \left[ -A^T H - H A - G \right] x(t) + e^{\gamma t} \left( 1 - e^{-\beta \tau} \right) x^T (t - \tau) G x(t - \tau). \quad (11) \]

We suppose, as follows from conditions of the theorem 1, there are positively determined matrixes \( G \) and \( H \) at which the matrix \( S [G, H] \) also is positively determined and \( \beta \geq \gamma \geq 0 \). Then we obtain

\[ \frac{d}{dt} V [x(t), t] \leq -e^{\gamma t} \left\{ \lambda_{\min} (S [G, H]) - \gamma \lambda_{\max} (H) \right\} \| x(t) \|^2 - \]
\[-e^{\gamma t} \left\{ \lambda_{\min} (S [G, H]) - (1 - e^{-\beta \tau}) \lambda_{\max}(G) \right\} |x(t - \tau)|^2 -
\]
\[-e^{\gamma t} (\beta - \gamma) \lambda_{\min} (G) \|x(t)\|_{\tau,\beta}^2 \]  \hspace{1cm} (12)

If the parameters of system and functional are
\[\lambda_{\min} (S [G, H]) \geq \lambda_{\max}(G)\]
than from an inequality (12) follows, that
\[
\frac{d}{dt} V [x(t), t] \leq -e^{\gamma t} \left\{ \lambda_{\min} (S [G, H]) - \gamma \lambda_{\max}(H) \right\} |x(t)|^2 -
\]
\[-e^{\gamma t} (\beta - \gamma) \lambda_{\min} (G) \|x(t)\|_{\tau,\beta}^2 \]  \hspace{1cm} (13)

for any \( \beta \geq 0 \).

If
\[\lambda_{\min} (S [G, H]) < \lambda_{\max}(G),\]
that inequality (13) will be carried out for
\[0 \leq \beta < \frac{1}{\tau} \ln \left[ \frac{\lambda_{\max}(G)}{\lambda_{\max}(G) - \lambda_{\min}(S [G, H])} \right].\]

We’ll transform right part of the inequality of square-law forms (9) to

\[-e^{\gamma t} \lambda_{\max}(H) |x(t)|^2 - e^{\gamma t} \lambda_{\max}(G) \|x(t)\|_{\tau,\beta}^2 \leq -V [x(t), t] \]  \hspace{1cm} (14)

Let’s consider two cases.
Let’s transform the inequality (14) as

\[-e^{\gamma t} |x(t)|^2 \leq -\frac{1}{\lambda_{\max}(H)} V [x(t), t] + e^{\gamma t} \frac{\lambda_{\max}(G)}{\lambda_{\max}(H)} \|x(t)\|_{\tau,\beta}^2\]

and we’ll substitute it in the first composed of inequalities (13). We get

\[
\frac{d}{dt} V [x(t), t] \leq -\frac{\lambda_{\min} (S [G, H]) - \gamma \lambda_{\max}(H)}{\lambda_{\max}(H)} V [x(t), t] -
\]
\[-e^{\gamma t} \left\{ (\beta - \gamma) \lambda_{\min} (G) - [\lambda_{\min} (S [G, H]) - \gamma \lambda_{\max}(H)] \frac{\lambda_{\max}(G)}{\lambda_{\max}(H)} \right\} \|x(t)\|_{\tau,\beta}^2.\]
If the parameters are
\[
(\beta - \gamma)\lambda_{\min}(G) \geq [\lambda_{\min}(S[G, H]) - \gamma\lambda_{\max}(H)] \frac{\lambda_{\max}(G)}{\lambda_{\max}(H)},
\]  
(15)
then
\[
\frac{d}{dt} V[x(t), t] \leq -\frac{\lambda_{\min}(S[G, H]) - \gamma\lambda_{\max}(H)}{\lambda_{\max}(H)} V[x(t), t].
\]
Solving the received differential inequality, we obtain
\[
V[x(t), t] \leq V[x(0), 0] e^{-\alpha t},
\]
\[
\alpha = \frac{\lambda_{\min}(S[G, H]) - \gamma\lambda_{\max}(H)}{\lambda_{\max}(H)}, \quad t \geq 0.
\]  
(16)
From here
\[
\zeta = \alpha + \gamma = \frac{\lambda_{\min}(S[G, H])}{\lambda_{\max}(H)}
\]
We’ll transform an inequality (14) to the following view
\[
-e^{\gamma t} ||x(t)||_{\tau,\beta} \leq -\frac{1}{\lambda_{\max}(G)} V[x(t), t] + e^{\gamma t} \frac{\lambda_{\max}(H)}{\lambda_{\max}(G)} |x(t)|^2
\]
and again we’ll substitute it in the second composed of inequalities (13). We receive
\[
\frac{d}{dt} V[x(t), t] \leq -(\beta - \gamma) \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} V[x(t), t] -
\]
\[
-e^{\gamma t} \left\{ \lambda_{\min}(S[G, H]) - \gamma\lambda_{\max}(H) - (\beta - \gamma) \lambda_{\min}(G) \frac{\lambda_{\max}(H)}{\lambda_{\max}(G)} \right\} |x(t)|^2
\]
And, if parameters are
\[
\lambda_{\min}(S[G, H]) - \gamma\lambda_{\max}(H) - (\beta - \gamma) \lambda_{\min}(G) \frac{\lambda_{\max}(H)}{\lambda_{\max}(G)} > 0,
\]  
(17)
then
\[
\frac{d}{dt} V[x(t)] \leq -(\beta - \gamma) \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} V[x(t)].
\]
Having integrated the received expression, we’ll obtain
\[ V[x(t), t] \leq V[x(0), 0] e^{-\alpha t}, \]

\[ \alpha = (\beta - \gamma) \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}, \quad t \geq 0. \quad (18) \]

From here we have

\[ \zeta = \alpha + \gamma = \beta \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} + \gamma \left[ 1 - \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} \right] \]

For reception of required result we’ll return to bilateral estimations of Lyapunov - Krasovsky functional (9). Using expressions (16), (18), we shall write down

\[ e^{\alpha t} \left\{ \lambda_{\min}(H) |x(t)|^2 + \lambda_{\min}(G) \|x(t)\|_{\tau, \beta}^2 \right\} \leq \]

\[ \leq V[x(t), t] \leq V[x(0), 0] e^{-\alpha t} \leq \]

\[ \leq e^{-\alpha t} \left\{ \lambda_{\max}(H) |x(0)|^2 + \lambda_{\max}(G) \|x(0)\|_{\tau, \beta}^2 \right\} \]

From here it is possible to obtain two estimations. First, we get

\[ |x(t)|^2 \leq \left[ \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)} |x(0)|^2 + \frac{\lambda_{\max}(G)}{\lambda_{\min}(H)} \|x(0)\|_{\tau, \beta}^2 \right] e^{-(\alpha + \gamma)t}. \]

And, using designations \varphi_{11}(H), \varphi_{12}(G, H), we receive

\[ |x(t)| \leq \left[ \sqrt{\varphi_{11}(H)} |x(0)| + \sqrt{\varphi_{12}(G, H)} \|x(0)\|_{\tau, \beta} \right] \exp \left\{ -\frac{1}{2} (\alpha + \gamma) t \right\}, \quad t \geq 0, \]

Further it is possible to write down

\[ \|x(t)\|_{\tau, \beta}^2 \leq \left[ \frac{\lambda_{\max}(H)}{\lambda_{\min}(G)} |x(0)|^2 + \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)} \|x(0)\|_{\tau, \beta}^2 \right] e^{-(\alpha + \gamma)t}. \]

And, using designations \varphi_{21}(G, H), \varphi_{22}(G), we receive an inequality

\[ \|x(t)\|_{\tau, \beta} \leq \left[ \sqrt{\varphi_{21}(G, H)} |x(0)| + \sqrt{\varphi_{22}(G)} \|x(0)\|_{\tau, \beta} \right] \exp \left\{ -\frac{1}{2} \varsigma t \right\}, \quad t \geq 0. \]
As follows from consideration of both cases we have

$$\varsigma = \frac{\lambda_{\text{min}}(S[G,H])}{\lambda_{\text{max}}(H)}$$

(19)

for \(\beta \frac{\lambda_{\text{min}}(G)}{\lambda_{\text{max}}(G)} + \gamma \left[1 - \frac{\lambda_{\text{min}}(G)}{\lambda_{\text{max}}(G)}\right] \geq \frac{\lambda_{\text{min}}(S[G,H])}{\lambda_{\text{max}}(H)}\)

$$\varsigma = \frac{\beta \lambda_{\text{min}}(G)}{\lambda_{\text{max}}(G)} + \gamma \left[1 - \frac{\lambda_{\text{min}}(G)}{\lambda_{\text{max}}(G)}\right]$$

(20)

for \(\beta \frac{\lambda_{\text{min}}(G)}{\lambda_{\text{max}}(G)} + \gamma \left[1 - \frac{\lambda_{\text{min}}(G)}{\lambda_{\text{max}}(G)}\right] < \frac{\lambda_{\text{min}}(S[G,H])}{\lambda_{\text{max}}(H)}\)

Uniting these expressions, we obtain the statement of the theorem 1.

We shall consider a case when it is not possible to find a matrix \(G\) and \(H\) at which the matrix \(S[G,H]\) is positively determined.

Let’s denote

$$S[G,H,\gamma] = \begin{bmatrix} -A^T H - HA - \gamma H - G & -HB \\ -B^T H & G \end{bmatrix}. \tag{21}$$

Obviously, due to a choice of a scalar value \(\gamma < 0\) the matrix \(S[G,H,\gamma]\) can be made positively certain.

**Lemma.** Let the matrixes \(G, H\) are positively determined and such inequality also is carried out

$$\gamma < \frac{\lambda_{\text{min}}[-A^T H - HA - G - HBG^{-1}B^T H]}{\lambda_{\text{max}}(H)}$$

(22)

Then the matrix \(S[G,H,\gamma]\) also is positively determined.

Using the lemma, we obtain the following result.

**Theorem 2.** Let a positively certain matrixes \(G, H\) at which the matrix \(S[G,H]\) also is positively determined are not exist. If the value \(\gamma\) is chosen according to an inequality (22) and \(\beta \geq \gamma\) then for the solutions \(x(t)\) of system (3) are fair top exponential estimations of convergence (6), (7).

$$|x(t)| \leq \left[\sqrt{\varphi_{11}(H)}|x(0)| + \sqrt{\varphi_{12}(G,H)}\|x(0)\|_{\tau,\beta}\right] \exp\left\{-\frac{1}{2} \varsigma t\right\},$$

\(t \geq 0,\)
\[ \|x(t)\|_{\tau,\beta} \leq \left[ \sqrt{\varphi_{21}(G \| H)} |x(0)| + \sqrt{\varphi_{22}(G)} \|x(0)\|_{\tau,\beta} \right] \exp \left\{ -\frac{1}{2} \varsigma t \right\}, \]

\[ t \geq 0 \]

\[ \varsigma (\beta, \gamma) = \min \left\{ \frac{\lambda_{\min} (S [G, H])}{\lambda_{\max} (H)} + \gamma, \beta \frac{\lambda_{\min} (G)}{\lambda_{\max} (G)} + \gamma \left[ 1 - \frac{\lambda_{\min} (G)}{\lambda_{\max} (G)} \right] \right\}, \quad (23) \]

The value \( \beta \) can be arbitrary if

\[ \lambda_{\min} (S [G, H, \gamma]) \geq \lambda_{\max} (G) \]

And

\[ \beta \leq \frac{1}{\tau} \ln \left\{ \frac{\lambda_{\max} (G)}{\lambda_{\max} (G) - \lambda_{\min} (S [G, H, \gamma])} \right\}, \]

if

\[ \lambda_{\min} (S [G, H, \gamma]) < \lambda_{\max} (H). \]

The proof to be conducted analogically to theorem 1.

**Remark 1.** As for value \( \|x(t)\|_{\tau,\beta}^2 \) the top estimations are fair

\[ \|x(t)\|_{\tau,\beta}^2 = \int_{-\tau}^{0} e^{\beta s} |x(t+s)| ds \leq \max_{-\tau \leq s \leq 0} \{|x(t+s)|^2\} \int_{-\tau}^{0} e^{\beta s} ds \leq \frac{1}{\beta} \left( 1 - e^{-\beta \tau} \right) \|x(t)\|_{\tau}^2 \leq \tau \|x(t)\|_{\tau} \]

where

\[ \|x(t)\|_{\tau} = \max_{-\tau \leq s \leq 0} \{|x(t+s)|\}, \]

than it is possible the inequality (6) transforms to following

\[ |x(t)| \leq \left[ \sqrt{\varphi_{11}(H)} |x(0)| + \sqrt{\varphi_{12}(G \| H)} \|x(0)\|_{\tau} \right] e^{-\frac{1}{2} \varsigma t}, t \geq 0, \]

or, even,
\[ |x(t)| \leq \left[ \sqrt{\varphi_{11}(H)} + \sqrt{\varphi_{12}(G, H)} \right] \|x(0)\| e^{-\frac{1}{2}\tau t}, \quad t \geq 0, \quad (24) \]

**Remark 2.** As estimations of majorant type than they contain two free parameters \( \beta \) and \( \gamma \), and in the second theorem \( \gamma \) can be negative. If to put a task of a finding of “an optimum estimation” in given class of functionals, the parameters \( \beta \) and \( \gamma \) it is possible to calculate precisely.

**Main Result**

In the previous sections majorant estimations of solutions of stable and unstable subsystems are separately received. now we shall consider whole hybrid system (1).

Let on each of intervals \( t_{i-1} \leq t < t_i, \quad i = \overline{1, N} \) for research Lyapunov - Krasovskiy functional of a kind (4) with positively certain matrices \( H_i, G_i, \quad i = \overline{1, N} \) is chosen. If there are positively certain matrices \( H_i, G_i, \quad i \in I \), such that matrices

\[
S_i [G_i, H_i] = \begin{bmatrix}
-A_i^T H_i - H_i A_i - G_i & -H_i B_i \\
-B_i^T H_i & G_i
\end{bmatrix}, \quad i \in I
\]

are positively certain, than we designate

\[
N_i = \left[ \sqrt{\varphi_{11}(H_i)} + \sqrt{\varphi_{12}(G_i, H_i)} \right] \exp \{ \varsigma_i (\beta_i, \gamma_i) \tau \},
\]

where value \( \beta_i > 0 \) can be arbitrary at

\[
\lambda_{\min} (S [G_i, H_i]) \geq \lambda_{\max} (G_i)
\]

and

\[
\beta_i \leq \frac{1}{\tau} \ln \left\{ \frac{\lambda_{\max} (G_i)}{\lambda_{\max} (G_i) - \lambda_{\min} (S [G_i, H_i])} \right\},
\]

if

\[
\lambda_{\min} (S [G_i, H_i]) < \lambda_{\max} (H_i).
\]

The value \( \gamma \) satisfied to a condition \( \gamma \leq \beta \).

If such matrices \( H_i, G_i, \quad i \in \overline{1, N} \) are not exist than we assume

\[
\gamma_j < \frac{\lambda_{\min} \left[ -A_j^T H_j - H_j A_j - G_j - H_j B_j G_j^{-1} B_j^T H_j \right]}{\lambda_{\max} (H_j)},
\]

and we designate
\[ S [G_j, H_j, \gamma_j] = \begin{bmatrix} -A_j^T H_j - H_j A_j - \gamma_j H_j - G_j - H_j B_j \\ -B_j^T H_j \end{bmatrix}, \]

\[ N_j = \left[ \sqrt{\phi_{11} (H_j)} + \sqrt{\phi_{12} (G_j, H_j)} \right] \exp \{ \varsigma_j (\beta_j, \gamma_j) \}, \]

where

\[ \varsigma_j (\beta_j, \gamma_j) = \min \left\{ \frac{\lambda_{\min} (S [G_j, H_j, \gamma_j])}{\lambda_{\max} (H_j)} + \gamma_j, \beta_j \frac{\lambda_{\min} (G_j)}{\lambda_{\max} (G_j)} + \gamma_j \left[ 1 - \frac{\lambda_{\min} (G_j)}{\lambda_{\max} (G_j)} \right] \right\} \]

The value \( \beta_j \) can be arbitrary at

\[ \lambda_{\min} (S [G_j, H_j, \gamma_j]) \geq \lambda_{\max} (G_j) \]

and

\[ \beta_j \leq \frac{1}{\tau} \ln \left\{ \frac{\lambda_{\max} (G_j)}{\lambda_{\max} (G_j) - \lambda_{\min} (S [G_j, H_j, \gamma_j])} \right\}, \]

if

\[ \lambda_{\min} (S [G_j, H_j, \gamma_j]) < \lambda_{\max} (H_j). \]

**Theorem 3.** Let the initial state of the logic-dynamical Hybrid system (1) satisfy to the condition \( \| x(0) \|_\tau < \delta \). Then at \( t = t_N \) inequality will be executed

\[ \| x(t_N) \| \leq \prod_{i=1}^N N_i \exp \left\{ -\frac{1}{2} \sum_{i=1}^N \varsigma_i (t_i - t_{i-1}) \right\}. \]

**Proof.** Let consider the first time interval \( t_0 \leq t \leq t_1, \ t_0 = 0 \). If there are positively certain matrices \( G_1, H_1 \), at which matrix \( S [G_1, H_1] \) also positively certain, than as follows from expression (24) of remark 1, inequality is executed

\[ \| x(t_1) \| \leq \left[ \sqrt{\phi_1 (H_1)} + \phi (G_1, H_1) \right] \| x(t_0) \|_\tau e^{-\frac{1}{2} \varsigma_1 (t_1 - \tau)}. \]

If there are not such matrices, for arbitrary positively certain matrices \( G_1, H_1 \), there exists a constant \( \gamma_1 \), at which a matrix \( S [G_1, H_1, \gamma_1] \) also will be positively certain. Again using expression (24) of remark 1, we get

\[ \| x(t_1) \| \leq \left[ \sqrt{\phi_1 (H_1)} + \phi (G_1, H_1) \right] \| x(t_0) \|_\tau e^{-\frac{1}{2} \varsigma_1 (t_1 - t_0)}. \]
And for the time moment \( t = t_1 \) takes place
\[
\|x(t_1)\|_\tau \leq N_1 \|x(t_0)\|_\tau e^{-\frac{1}{2} \varsigma_1 (t_1 - t_0)}.
\]

Let we consider a next interval \( t_1 \leq t \leq t_2 \). As for the second interval a similar estimation takes place
\[
\|x(t_2)\|_\tau \leq N_2 \|x(t_1)\|_\tau e^{-\frac{1}{2} \varsigma_2 (t_2 - t_1)},
\]
we obtain
\[
\|x(t_2)\|_\tau \leq N_1 N_2 \|x(t_0)\|_\tau \exp \left\{ -\frac{1}{2} \left[ \varsigma_1 (t_1 - t_0) + \varsigma_2 (t_2 - t_1) \right] \right\}.
\]

Continuing a process farther, for the time moment \( t = t_N \) we get
\[
\|x(t_N)\| \leq \prod_{i=1}^{N} N_i \exp \left\{ -\frac{1}{2} \sum_{i=1}^{N} \varsigma_i (t_i - t_{i-1}) \right\},
\]
that it was required to prove.

**Literature**


