Controllability, autonomicity and free variables in periodic behaviors

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Abstract

In this paper we propose a new type of “image” representation for periodic behaviors and analyze the relationship between image representability and controllability, showing that the correspondence, which exists in the time-invariant case, generalizes to periodic behaviors. We also show that the concept of free variable used in the time-invariant case cannot be transposed to the periodic case in a straightforward manner and introduce a new concept of variable freeness (P-periodic freeness). This allows us to define input/output structures for periodic behaviors.

1 Introduction

In this paper we present new results on the behavioral theory of linear periodic systems and analyze the resemblances and differences with the time-invariant case. The paper builds on a framework that was developed Kuijper and Willems, see [2] and the references therein. An important tool is the lifting technique that associates to each periodic behavior a time-invariant behavior. Since it is possible to relate some system theoretic properties of a periodic behavior to similar properties of the corresponding lifted behavior, this allows to use results for the time-invariant case and, subsequently, translate them back into the periodic case. Using this technique, we have presented in [1] some further insights into (what we called) “kernel representations”, controllability and autonomicity for periodic behaviors, as well as the characterization of these properties in terms of those representations. Here we propose a definition of “image representation” for periodic behaviors and show the correspondence between controllability and the existence of image representations. This is well-understood in the time-invariant behavioral theory, and turns out to be valid in the periodic case as well. In spite of all the formal resemblances, there are some fundamental differences between time-invariant and periodic behaviors. This is, for instance, the case with the relationship between free variables, controllability and autonomicity. A naive definition of free variables would lead to the situation that a (nonzero) periodic behavior without free variables is not necessarily autonomous and may even be controllable. In order to overcome this situation we introduce a new concept of periodically free variable, which also allows us to define inputs and outputs in a periodic system.
We focus our attention on the discrete time case, i.e., on behaviors $\mathcal{B}$ whose trajectories belong to $(\mathbb{R}^q)^Z$. As is well known, the behavior of a time-invariant system is characterized by its invariance under the time shift, i.e.,

$$\sigma \mathcal{B} = \mathcal{B},$$

where $\sigma^\lambda : (\mathbb{R}^q)^Z \to (\mathbb{R}^q)^Z$, defined by

$$\left(\sigma^\lambda w\right)(k) := w(k+\lambda)$$

is called backward $\lambda$-shift in case $\lambda \in \mathbb{Z}_+$ or forward $\lambda$-shift in case $\lambda \in \mathbb{Z}_-$. Periodic behaviors are characterized by their invariance with respect to the $P$-shift ($P \in \mathbb{N}$). More concretely a behavior $\mathcal{B}$ is said to be $P$-periodic if

$$\sigma^P \mathcal{B} = \mathcal{B},$$

i.e., if $\mathcal{B}$ is invariant with respect to the $P$-shift, but not with respect to any $\lambda$-shift with $0 < \lambda < P$.

An efficient way of dealing with a periodic system is to relate it with a suitable time-invariant one.

Here, following the approach introduced in [2], we associate with a $P$-periodic behavior $\mathcal{B} \subset (\mathbb{R}^q)^Z$, a time-invariant behavior $L \mathcal{B} \subset (\mathbb{R}^Pq)^Z$, the lifted-behavior, defined by

$$L \mathcal{B} = L(\mathcal{B}) := \left\{\tilde{w} \in (\mathbb{R}^Pq)^Z \mid \tilde{w} = Lw, \ w \in \mathcal{B}\right\},$$

where $L$ is the linear map

$$L : (\mathbb{R}^q)^Z \to (\mathbb{R}^Pq)^Z,$$

defined by

$$(Lw)(k) := \begin{bmatrix}
w(Pk+1) \\
\vdots \\
w(Pk+P)
\end{bmatrix}.$$

We shall consider the class of $P$-periodic behaviors which constitute closed linear subspaces of $(\mathbb{R}^q)^Z$ (in the topology of pointwise convergence). Henceforth we shall refer to these behaviors simply as $P$-periodic behaviors. As shown in [2] and [4], such behaviors can be described by dynamical equations of the form:

$$\left(R_t \left(\sigma, \sigma^{-1}\right) w\right)(Pk+t) = 0, \ t = 1, \ldots, P, \ k \in \mathbb{Z}, \quad (1)$$

where $R_t (\xi, \xi^{-1}) \in \mathbb{R}^{g \times q} [\xi, \xi^{-1}]$. Analogously to the time-invariant case, although with some abuse of language, we refer to (1) as a $P$-periodic kernel representation (PPKR). Note that, since

$$\left(R_t \left(\sigma, \sigma^{-1}\right) w\right)(Pk+t) = \left(\sigma^t R_t \left(\sigma, \sigma^{-1}\right)\right) w(Pk), \ t = 1, \ldots, P, \ k \in \mathbb{Z},$$

the $P$-periodic kernel representation (1) can be written as

$$\left(R \left(\sigma, \sigma^{-1}\right) w\right)(Pk) = 0, \ k \in \mathbb{Z}, \quad (2)$$
where

\[
R(\xi,\xi^{-1}) := \begin{bmatrix}
\xi R_1(\xi,\xi^{-1}) \\
\xi^2 R_2(\xi,\xi^{-1}) \\
\vdots \\
\xi^P R_P(\xi,\xi^{-1})
\end{bmatrix} \in \mathbb{R}^{g \times q} [\xi,\xi^{-1}],
\]

with \( g := \sum_{t=1}^P g_t \). From now on we refer to the matrix \( R(\xi,\xi^{-1}) \) as a PPKR matrix of the corresponding behavior.

Now, decompose \( R(\xi,\xi^{-1}) \) as

\[
R(\xi,\xi^{-1}) = \xi R_1(\xi,\xi^{-1}) + \cdots + \xi^P R_P(\xi,\xi^{-1}) = R^L(\xi,\xi^{-1}) \Omega_{P,q}(\xi),
\]

with

\[
\Omega_{P,q}(\xi) := \begin{bmatrix}
\xi I_q & \cdots & \xi^P I_q
\end{bmatrix}^T
\]

(3)

and

\[
R^L(\xi,\xi^{-1}) = \begin{bmatrix}
R_1(\xi,\xi^{-1}) & R_2(\xi,\xi^{-1}) & \cdots & R_P(\xi,\xi^{-1})
\end{bmatrix}.
\]

(4)

Recalling the definition of the lifted trajectory \( Lw \) associated to \( w \), (2) can be written as

\[
(R^L(\sigma,\sigma^{-1})(Lw))(k) = 0, \quad k \in \mathbb{Z}.
\]

This allows us to conclude that the lifted behavior \( L\mathcal{B} \) is given by the kernel representation

\[
L\mathcal{B} = \{ \tilde{w} | R^L(\sigma,\sigma^{-1})\tilde{w} = 0 \} = \ker R^L(\sigma,\sigma^{-1}).
\]

3 P-periodic image representations

Image representations, that yield a behavior, not as the kernel, but as the image of a shift-operator, constitute an important type of description in the time-invariant case. As a generalization of such representations we introduce here \( P \)-periodic image representations (PPIR).

**Definition 3.1** A behavior \( \mathcal{B} \) is said to have a \( P \)-periodic image representation (PPIR) if it can be described by equations of the form:

\[
w(Pk+t) = (M_t(\sigma,\sigma^{-1})v)(Pk+t), \quad t = 1, \ldots, P, \quad k \in \mathbb{Z},
\]

(5)

where \( w \in (\mathbb{R}^q)_{\mathbb{Z}} \) is the system variable and \( v \) is an auxiliary variable taking values in \( \mathbb{R}^\ell, \ell \in \mathbb{N} \).

Similar to what was done for the PPKR case, (5) can be written as

\[
\begin{bmatrix}
w(Pk+1) \\
w(Pk+2) \\
\vdots \\
w(Pk+P)
\end{bmatrix} = (M(\sigma,\sigma^{-1})v)(Pk), \quad k \in \mathbb{Z},
\]

where

\[
M(\sigma,\sigma^{-1}) := \begin{bmatrix}
M_1(\sigma,\sigma^{-1}) \\
M_2(\sigma,\sigma^{-1}) \\
\vdots \\
M_P(\sigma,\sigma^{-1})
\end{bmatrix} \in \mathbb{R}^{\ell \times q} [\sigma,\sigma^{-1}].
\]
with
\[
M (\xi, \xi^{-1}) := \begin{bmatrix}
\xi M_1 (\xi, \xi^{-1}) \\
\xi^2 M_2 (\xi, \xi^{-1}) \\
\vdots \\
\xi^P M_P (\xi, \xi^{-1})
\end{bmatrix} \in \mathbb{R}^{Pq \times \ell} [\xi; \xi^{-1}];
\]
we refer to this matrix as a \textit{PPIR matrix}. This leads to
\[
(Lw) (k) = (M^L (\sigma, \sigma^{-1}) (Lv)) (k), \ k \in \mathbb{Z},
\]
where \( M^L \) is such that
\[
M (\xi, \xi^{-1}) = M^L (\xi^P, \xi^{-P}) \Omega_{P, \ell} (\xi).
\]
(6)

Therefore, if \( \mathcal{B} \) is a \( P \)-periodic behavior with \textit{PPIR matrix} \( M \), \( L\mathcal{B} \) is a time-invariant behavior with image representation \( M^L \). It turns out that the converse also holds true.

\textbf{Theorem 3.2} A \( P \)-periodic behavior \( \mathcal{B} \subset (\mathbb{R}^q)^\mathbb{Z} \) has an image representation if and only if the associated lifted behavior \( L\mathcal{B} \) has an image representation.

\textbf{Example 3.3} Consider the 2-periodic behavior \( \mathcal{B} \) with \textit{PPKR}
\[
R (\xi, \xi^{-1}) = \xi - 1.
\]
Since
\[
R (\xi, \xi^{-1}) = \xi - 1 = \begin{bmatrix} 1 & -\xi^2 \end{bmatrix} \begin{bmatrix} \xi \\ \xi^2 \end{bmatrix},
\]
it its associated lifted behavior \( L\mathcal{B} \) is described by the kernel representation
\[
\left( R^L (\sigma, \sigma^{-1}) \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{bmatrix} \right) (k) = 0, \ k \in \mathbb{Z},
\]
where
\[
R^L (\xi, \xi^{-1}) = \begin{bmatrix} 1 & -\xi^{-1} \end{bmatrix}.
\]
It is also possible to describe this lifted behavior in terms of an image representation, namely
\[
L\mathcal{B} = \text{im} \begin{bmatrix} 1 \\ \sigma \end{bmatrix}.
\]
In order to achieve equation (6) requirement we consider that
\[
L\mathcal{B} = \text{im} M^L (\sigma, \sigma^{-1}),
\]
with \( M^L (\xi, \xi^{-1}) \in \mathbb{R}^{2 \times 2\ell} [\xi; \xi^{-1}] \), given by
\[
M^L (\xi, \xi^{-1}) = \begin{bmatrix} 1 & 0 \\ \xi & 0 \end{bmatrix}.
\]
Therefore the original 2-periodic behavior has a PPIR matrix $M$ given by

$$M (\xi, \xi^{-1}) = \begin{bmatrix} 1 & 0 \\ \xi^2 & 0 \end{bmatrix} \Omega_{2,1} (\xi)$$

$$= \begin{bmatrix} 1 & 0 \\ \xi^2 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \xi^2 \end{bmatrix}$$

$$= \begin{bmatrix} \xi \\ \xi^3 \end{bmatrix},$$

that is, the 2-periodic behavior $\mathcal{B}$ allows the image representation

$$\mathcal{B} = \left\{ w \in (\mathbb{R})^\mathbb{Z} : \exists v \in (\mathbb{R})^\mathbb{Z} \text{ s.t. } (7) \text{ holds} \right\},$$

with

$$\begin{bmatrix} w (2k + 1) \\ w (2k + 2) \end{bmatrix} = (M (\sigma, \sigma^{-1}) v) (2k), \; k \in \mathbb{Z}. \tag{7}$$

## 4 Controllability

According to the definition of behavioral controllability, a behavior $\mathcal{B}$ is said to be *controllable* if the past of every trajectory in $\mathcal{B}$ can be concatenated with the future of an arbitrary trajectory in this behavior. More concretely,

**Definition 4.1** [3] A behavior $\mathcal{B}$ is said to be controllable if for all $w_1, \; w_2 \in \mathcal{B}$ and $k_0 \in \mathbb{Z}$, there exists $k_1 \geq 0$ and $w \in \mathcal{B}$ such that $w (k) = w_1 (k)$, for $k \leq k_0$, and $w (k) = w_2 (k)$, for $k > k_0 + k_1$.

As shown in [1]:

**Theorem 4.2** A $P$-periodic behavior $\mathcal{B}$ is controllable if and only if the corresponding lifted behavior $L\mathcal{B}$ is controllable.

Due to the well-know equivalence between behavioral controllability and the existence of image representations for time-invariant behaviors (see [3]), this theorem, together with Theorem 3.2, yields the following result.

**Theorem 4.3** A $P$-periodic behavior $\mathcal{B}$ has a PPIR if and only if it is controllable.

**Example 4.4** Consider the 2-periodic behavior $\mathcal{B}$ of Example 3.3. The associated lifted behavior $L\mathcal{B}$ has an image representation and consequently is controllable. Finally the controllability of $L\mathcal{B}$ leads to the controllability of $\mathcal{B}$. 

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5 Autonomicity

As explained, controllability, roughly speaking, is the possibility of changing from one system trajectory to any other one. At the other end of the spectrum stands autonomicity. In an autonomous behavior every trajectory is uniquely determined by its past.

**Definition 5.1** A $P$-periodic behavior $\mathcal{B}$ is said to be autonomous if for all $k_0 \in \mathbb{Z}$ and all $w_1, w_2 \in \mathcal{B}$

$$w_1(k) = w_2(k) \text{ for } k < k_0 \implies w_1 = w_2.$$  

Similar to what is the case with controllability, the autonomicity of $\mathcal{B}$ and of $L\mathcal{B}$ are one-to-one related.

**Theorem 5.2** [2] Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B})$ be a $P$-periodic system. Then $\mathcal{B}$ is autonomous if and only if $L\mathcal{B}$ is autonomous.

Since, a $P$-periodic behavior $\mathcal{B}$ with $\text{PPKR } R(\xi, \xi^{-1})$ is associated with the lifted time-invariant behavior $L\mathcal{B} = \ker R^L(\sigma, \sigma^{-1})$, where $R^L$ is obtained as in (4), and taking into account the characterization of autonomicity for time-invariant behaviors given in [3], the following result is trivially obtained.

**Corollary 5.3** Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B})$ be a $P$-periodic system with a $\text{PPKR and a representation matrix } R$. Then $\mathcal{B}$ is autonomous if and only if the corresponding representation matrix of the associated lifted system, $R^L$, has full column rank (fcr).

6 Free variables

Recall that given a behavior $\mathcal{B} \subset (\mathbb{R}^q)^\mathbb{Z}$, a component $w_i$ of the system variable $w$ is said to be free if for all $\alpha \in \mathbb{R}^\mathbb{Z}$ there exist a trajectory $w^* \in \mathcal{B}$ such that $w^*_i(k) = \alpha(k), k \in \mathbb{Z}$. In other words it is not restricted by the system laws.

In the time-invariant case, the existence or absence of free variables is related to proprieties as controllability and autonomicity. Indeed, a non-trivial time-invariant controllable behavior must have free variables. On the other hand the absence of free variables is equivalent to autonomicity, [3]. As the next examples shows, this no longer holds in the $P$-periodic case.

**Example 6.1** Consider again the 2-periodic behavior $\mathcal{B}$ of Example 3.3, with $\text{PPKR } R(\xi, \xi^{-1}) = \xi - 1$, i.e., described by

$$w(2k + 1) = w(2k), \quad k \in \mathbb{Z}.$$  

As shown in Example 4.4, $\mathcal{B}$ is controllable since the associated lifted behavior, $L\mathcal{B} = \text{im } \begin{bmatrix} 1 \\ \sigma \end{bmatrix}$, is. However $\mathcal{B}$ has no free variables, since the values of $w$ on each even time instant and its consecutive one must coincide.
Example 6.2 Let \( B \subset \mathbb{R}^{Z} \) be the 2-periodic behavior described by \( w(2k) = 0, \ k \in \mathbb{Z} \). Clearly the only system variable \( w \) is not free, since it is required to be zero on even time instants. However, \( B \) is not autonomous. Indeed fixing the values of \( w(k) \) for \( k \leq 0 \) does not yield a unique trajectory, since the values of \( w(2k+1), \ k > 0 \) can still be chosen freely. Thus the absence of free variables does not imply autonomy.

These examples suggest that a different notion of free variable should be considered in the \( P \)-periodic case.

Definition 6.3 Let \( B \subset (\mathbb{R}^{q})^{Z} \) be a behavior in \( q \) variables. The \( i \)th system variable \( w_{i}, \ i \in \{1, \ldots, q\} \), is said to be \( P \)-periodically free with offset \( t \) or \( t-P \)-periodically free, for \( t = 1, \ldots, P \), if \( w_{i}(Pk + t), \ k \in \mathbb{Z}, \) is not restricted by the behavior. More precisely, if for all \( \alpha \in \mathbb{R}^{Z} \), there exists \( w^{*} \in B \) such that its \( i \)th-component satisfies

\[
w^{*}_{i}(Pk + t) = \alpha(k), \ k \in \mathbb{Z}.
\]

Moreover, \( w_{i} \) is said to be \( P \)-periodically free if it is \( P \)-periodically free with offset \( t \) for some \( t = 1, \ldots, P \).

Note that, regarding time-invariance as 1-periodicity, this definition yields the usual definition of free variable for time-invariant behaviors.

The following proposition is a direct consequence of this definition.

Proposition 6.4 Given a \( P \)-periodic behavior \( B \subset (\mathbb{R}^{q})^{Z} \), the \( i \)th system variable \( w_{i}, \ i \in \{1, \ldots, q\} \), is \( t-P \)-periodically free (in \( B \)) if and only if \( (Lw)_{(t-1)q+1} \) is free in \( LB \).

It is now not difficult to see that a controllable \( P \)-periodic behavior must have \( P \)-periodically free variables.

Example 6.5 Recall Examples 3.3, 4.4 and 6.1. As we have seen there, \( w \) is not free. However, we shall see now that this variable is 2-periodically free. Recall that the associated lifted behavior \( LB \) is described by

\[
\left( RL (\sigma, \sigma^{-1}) \left[ \begin{array}{c} \tilde{w}_{1} \\ \tilde{w}_{2} \end{array} \right] \right)(k) = 0, \ k \in \mathbb{Z},
\]

or, equivalently,

\[
\tilde{w}_{1}(k) = \tilde{w}_{2}(k-1), \ k \in \mathbb{Z},
\]

showing that either \( \tilde{w}_{1} \) or \( \tilde{w}_{2} \) are free in \( LB \). Thus \( w \) is 2-periodically free since it is 2-periodically free with offsets \( t = 1 \) or \( t = 2 \).

As for autonomicity, the following characterization in terms of \( P \)-periodically free variables holds.

Theorem 6.6 Let \( \Sigma = (Z, \mathbb{R}^{q}, B) \) be a \( P \)-periodic system. Then \( B \) is autonomous if and only if \( B \) has no \( P \)-periodically free variables.
Example 6.7 Recall Example 6.2. As we have seen there, although \( \mathcal{B} \) is not autonomous, the system variable \( w \) is not free. Remark that however \( w \) is 2-periodically free since we have in this case

\[
R(\xi, \xi^{-1}) = 1 = \begin{bmatrix} 0 & \xi^{-2} \\ \xi & \xi^2 \end{bmatrix},
\]

which leads to

\[
R^L(\xi, \xi^{-1}) = \begin{bmatrix} 0 & \xi^{-1} \end{bmatrix}.
\]

Therefore the associated lifted behavior \( L\mathcal{B} \) is described by

\[
\left( R^L(\sigma, \sigma^{-1}) \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{bmatrix} \right)(k) = 0, \ k \in \mathbb{Z},
\]

equivalently,

\[
\tilde{w}_2(k - 1) = 0, \ k \in \mathbb{Z},
\]

or still

\[
\tilde{w}_2(k) = 0, \ k \in \mathbb{Z}.
\]

Thus \( \tilde{w}_1 \) is free and \( w \) is 2-periodically free since it is 2-periodically free with offset \( t = 1 \).

In the time-invariant case, an input is defined as a maximally free set of system variables, i.e., as a set of variables which are simultaneously free and which, once fixed, leave no extra free variables in the system. When defining simultaneously free components of a signal in a \( P \)-periodic behavior, one has to take into account that such components may be \( P \)-periodically free with different offsets. This is illustrated in the following example.

Example 6.8 Let \( \mathcal{B} \subset (\mathbb{R}^2)^\mathbb{Z} \) be the 3-periodic behavior given by the equations

\[
w_2(3k + 1) = w_2(3k + 2) = w_1(3k + 3) = 0, \ k \in \mathbb{Z}.
\]

Clearly the values of \( w_1(3k + 1) \), \( w_1(3k + 2) \) and \( w_2(3k + 3) \) (\( k \in \mathbb{Z} \)) are free, i.e., \( w_1 \) is 3-periodically free with offsets 1 and 2, and \( w_2 \) is 3-periodically free with offset 3. Note further, that none of the variables is free at all the possible offsets \( t = 1, 2, 3 \).

Given a \( P \)-periodic behavior \( \mathcal{B} \) with variable \( w \), a choice (possibly repeated) of components of \( w \), \((w_{i_1} \cdots w_{i_m})^T, i_r \in \{1, \ldots, q\} \) for \( r = 1, \ldots, m \), is said to be \((t_1, \ldots, t_m)\)-\( P \)-periodically free, \( t_r \in \{1, \ldots, P\} \) for \( r = 1, \ldots, m \), if for all \( \alpha_r \in \mathbb{R}^\mathbb{Z} \), there exists \( w^* \in \mathcal{B} \) such that its \( i_r \)-th component satisfies

\[
w^*_{i_r}(Pk + t_r) = \alpha_r(k), \ k \in \mathbb{Z}.
\]

Note that \((w_{i_1} \cdots w_{i_m})^T \) is \((t_1, \ldots, t_m)\)-\( P \)-periodically free if and only if

\[
u = \left( (\Omega_{P,q}(\sigma)w)_{(t_1-1)q+i_1}, \ldots, (\Omega_{P,q}(\sigma)w)_{(t_m-1)q+i_m} \right)
\]

is a free set of variables of \( \Omega_{P,q}(\sigma)w \), with \( \Omega_{P,q}(\xi) \) defined as in (3).
Definition 6.9 Given a $P$-periodic behavior $\mathcal{B} \subset (\mathbb{R}^q)^\mathbb{Z}$ with variable $w = (w_1 \cdots w_q)^T$, a choice of components $u = \begin{bmatrix} \Omega_{P,q} \ldots \xi_3 & 2\xi_4 - \xi_3 \end{bmatrix}$.

Its associated lifted system has also a kernel representation, that is, $LB = \ker RL (\sigma, \sigma^{-1})$.

Proposition 6.11 It is obvious that:

Remark 6.10 The fact that $u = \begin{bmatrix} \Omega_{P,q} \ldots \xi_3 \end{bmatrix}$ is a $P$-periodic input means that the values of

$$w_{i_r} (Pk + t_r), \ i_r \in \{1, \ldots, q\}, \ r = 1, \ldots, m,$$

where $l_r = (t_r - 1)q + i_r$, may be freely assigned.

Given that

$$(\Omega_{P,q} (\sigma) w) (Pk) = (Lw) (k), \ k \in \mathbb{Z},$$

it is obvious that:

Proposition 6.11 $u = \begin{bmatrix} \Omega_{P,q} (\sigma) w \end{bmatrix}$ is a $P$-periodic input of $\mathcal{B}$ if and only if $\bar{u} = \begin{bmatrix} (Lw) \end{bmatrix}$ is an input of the time-invariant behavior $L\mathcal{B}$.

Given the relationship between the $P$-periodically free variables of a $P$-periodic behavior and the free variables of its associated lifted system, it is now possible to define input/output structures in the periodic case based on the available results for time-invariant systems. This leads to the following result.

Theorem 6.12 Every $P$-periodic behavior $\mathcal{B}$ admits an input/output structure.

Example 6.13 Consider the 3-periodic behavior $\mathcal{B}$ with PPKR matrix

$$R (\xi, \xi^{-1}) = \begin{bmatrix} \xi^2 - \xi & 2 \\ \xi^3 + \xi^2 & 2\xi^3 \\ \xi & \xi^3 - \xi^2 \\ 2\xi^4 + \xi^3 & 2\xi^4 - \xi^3 \end{bmatrix}.$$

Its associated lifted system has also a kernel representation, that is,

$$L\mathcal{B} = \ker R^L (\sigma, \sigma^{-1})$$.
with
\[
R^L (\xi, \xi^{-1}) = \begin{bmatrix}
-1 & 0 & 1 & 0 & 0 & 2\xi^{-1} \\
0 & 0 & 1 & 0 & 1 & 2 \\
1 & 0 & 0 & -1 & 0 & 1 \\
2\xi & 2\xi & 0 & 0 & 1 & -1 \\
\end{bmatrix}.
\]

Letting
\[
\tilde{R}^L (\xi, \xi^{-1}) = R^L (\xi, \xi^{-1})
\]
\[
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
\[
= \begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 2\xi^{-1} \\
0 & 1 & 0 & 1 & 0 & 2 \\
1 & 0 & -1 & 0 & 0 & 1 \\
2\xi & 0 & 0 & 1 & 2\xi & -1 \\
\end{bmatrix}
=: \begin{bmatrix}
P (\xi, \xi^{-1}) \\
Q (\xi, \xi^{-1})
\end{bmatrix},
\]

the lifted system can be represented as
\[
\left( P (\sigma, \sigma^{-1}) \begin{bmatrix}
(Lw)_1 \\
(Lw)_3 \\
(Lw)_4 \\
(Lw)_5 \\
\end{bmatrix}\right) (k) = \left( Q (\sigma, \sigma^{-1}) \begin{bmatrix}
(Lw)_2 \\
(Lw)_6 \\
\end{bmatrix}\right) (k), \ k \in \mathbb{Z}.
\]

Since \(\det P (\xi, \xi^{-1}) \neq 0, \tilde{u} := \begin{bmatrix} Lw_2 & Lw_6 \end{bmatrix}^T\) is an input in \(L\mathcal{B}\) and, consequently
\[
u = (\Omega_{3,2} (\sigma) w)_2, (\Omega_{3,2} (\sigma) w)_6)
\]
is a 3-periodic input for \(\mathcal{B}\).

### 7 Conclusion

In the sequel of the work carried out in [2] and [1], we have considered \(P\)-periodic systems within the framework of the behavioral approach. We defined a new type of representations, \(P\)-periodic image representations (PPIR), that generalize time-invariant image representations and related them with controllability. Moreover we have analyzed the relationship between the existence of free variables and controllability and autonomicity. It turns out that, contrary to what happens for time-invariant systems, nontrivial \(P\)-periodic behaviors without free variables are not necessarily autonomous and may even be controllable. This suggests that the usual concept of freedom is not the adequate one in the periodic case. Therefore we have introduced the notion of \(P\)-periodic free variable and, related to it, the notion of \(P\)-periodic input. In our opinion, these preliminary results will play an important role in other contexts, such as for instance the study of control problems for \(P\)-periodic behaviors.
References


