A Family of Fibonacci-like Sequences*

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Abstract – We consider the recurrence relation

$$G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} \alpha_j n^j,$$

where $G_0 = G_1 = 1$, and we express G_n in terms of the Fibonacci numbers F_n and F_{n-1}, and in the parameters $\alpha_0,...,\alpha_k$.

For integer values of k, $\alpha_0,...,\alpha_k$, the relation

$$G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} \alpha_j n^j, \qquad (1)$$

where $G_0 = G_1 = 1$, forms a difference equation that can be easily solved by standard methods. In this note we provide such a solution for equations of this type, in which we treat $\alpha_0,...,\alpha_k$ as parameters.

First, the solution $G_n^{(h)}$ of the corresponding homogeneous equation equals

$$G_n^{(h)} = C_1 \phi_1^n + C_2 \phi_2^n,$$

where $\phi_1 = ½(1+\sqrt{5})$ and $\phi_2 = ½(1-\sqrt{5})$; cf. e.g., [1] and [3].

Second, as a particular solution we try

$$G_n^{(p)} = \sum_{i=0}^{k} A_i n^i,$$

which yields

$$\sum_{i=0}^{k} A_i n^i - \sum_{i=0}^{k} A_i (n-1)^i - \sum_{i=0}^{k} A_i (n-2)^i - \sum_{i=0}^{k} \alpha_i n^i = 0$$

or

$$\sum_{i=0}^{k} A_i n^i - \sum_{i=0}^{k} \left[\sum_{l=0}^{i} A_i \binom{i}{l} (-1)^{i-l} (1+2^{i-l}) n^l \right] - \sum_{i=0}^{k} \alpha_i n^i = 0.$$

For each i $(0 \le i \le k)$, we have

$$A_i - \sum_{m=i}^{k} \beta_{im} A_m - \alpha_i = 0, \qquad (2)$$

where, for $m \ge i$,

$$\beta_{im} = \binom{m}{i} (-1)^{m-i} (1+2^{m-i}).$$

From the recurrence relation (2), $A_k,...,A_0$ can be computed (in that order): A_i is a linear combination of $\alpha_i,...,\alpha_k$. However, a more explicit expression for A_i can be obtained by setting

$$A_i = -\sum_{j=i}^{k} a_{ij} \alpha_j.$$

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(The minus sign happens to be convenient in the sequel). Then (2) implies

$$-\sum_{j=i}^{k} a_{ij}\alpha_j + \sum_{m=i}^{k}\beta_{im}\left(\sum_{l=m}^{k} a_{ml}\alpha_l\right) - \alpha_i = 0.$$

Since $\beta_{ii} = 2$, we have, for $0 \le i \le k$

$$a_{ii} = 1$$
$$a_{ij} = -\sum_{m=i+1}^{j}\beta_{im}a_{mj}, \qquad \text{if } j > i.$$

Hence

$$G_n^{(p)} = -\sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}\alpha_j n^i = -\sum_{j=0}^{k}\alpha_j\left(\sum_{i=0}^{j} a_{ij}n^i\right).$$

Finally, we ought to determine C_1 and C_2: $G_0 = G_1 = 1$ implies

$$C_1 + C_2 = 1 - G_0^{(p)}, \quad C_1\phi_1 + C_2\phi_2 = 1 - G_1^{(p)}.$$

These equalities yield

$$C_1 = ((G_0^{(p)} - 1)\phi_2 + 1 - G_1^{(p)})(\sqrt{5})^{-1}$$
$$= ((1 - G_0^{(p)})\phi_1 - G_1^{(p)} + G_0^{(p)})(\sqrt{5})^{-1},$$
$$C_2 = ((G_0^{(p)} - 1)\phi_1 + G_1^{(p)} - 1)(\sqrt{5})^{-1}$$
$$= -((1 - G_0^{(p)})\phi_2 - G_1^{(p)} + G_0^{(p)})(\sqrt{5})^{-1}$$

and

$$G_n = (1 - G_0^{(p)})F_n + (-G_1^{(p)} + G_0^{(p)})F_{n-1} + G_n^{(p)}.$$

j	$p_j(n)$
0	1
1	$n+3$
2	$n^2+6n+13$
3	$n^3+9n^2+39n+81$
4	$n^4+12n^3+78n^2+324n+673$
5	$n^5+15n^4+130n^3+810n^2+3365n+6993$
6	$n^6+18n^5+195n^4+1620n^3+10095n^2+41958n+87193$
7	$n^7+21n^6+273n^5+2835n^4+23555n^3+146853n^2+610351n+1268361$
8	$n^8+24n^7+364n^6+4536n^5+47110n^4+391608n^3+2441404n^2+10146888n+21086113$

Table 1.

Summarizing, we have the following proposition.

Proposition: *The solution of (1) can be expressed as*

$$G_n = (1 + \Lambda_k)F_n + \lambda_k F_{n-1} - \sum_{j=0}^{k}\alpha_j p_j(n)$$

where Λ_k *is a linear combination of* $\alpha_0,\ldots,\alpha_k$, λ_k *is a linear combination of* $\alpha_1,\ldots,\alpha_k$, *and for each j* ($0 \le j \le k$), $p_j(n)$ *is a polynomial of degree j:*

$$\Lambda_k = \sum_{j=0}^{k} a_{0j}\alpha_j\,, \qquad \lambda_k = \sum_{j=1}^{k}\left(\sum_{i=1}^{j} a_{ij}\right)\alpha_j\,, \qquad p_j(n) = \sum_{i=0}^{j} a_{ij}n^i.$$

Remarks:

(1) For $j = 0, 1, \ldots, 8$ the polynomials $p_j(n)$ are given in Table 1.

(2) No assumptions on $\alpha_0,\ldots,\alpha_k$ have been made; thus, they may be rational or real numbers as well.

(3) Changing $G_1 = 1$ into $G_1 = c$ only affects λ_k: it has to be increased with $c-1$.

(4) The coefficients of $\alpha_0, \alpha_1, \alpha_2, \ldots$ in Λ_k and of $\alpha_1, \alpha_2, \ldots$ in λ_k are independent of k. Thus they give rise to two infinite sequences Λ and λ of natural numbers, as k tends to infinity, of which the first few elements are

Λ: 1, 3, 13, 81, 673, 6993, 87193, 1268361, 21086113,...
λ: 1, 7, 49, 415, 4321, 53887, 783889, 13031935,...

Neither of these sequences is included in [2].

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References

1. C.L. Liu. *Introduction to Combinatorial Mathematics.* New York: McGraw-Hill, 1968.
2. N.J.A. Sloane. *A Handbook of Integer Sequences.* New York: Academic Press, 1973.
3. N.N. Vorobyov. *The Fibonacci Numbers.* Boston, Mass.: Heath, 1963.