Extensions of Language Families and Canonical Forms for Full AFL-structures*

Peter R.J. Asveld
Department of Applied Mathematics, Twente University of Technology
P.O. Box 217, 7500 AE Enschede, the Netherlands
e-mail: p.r.j.asveld@misc.utwente.nl P.R.J.Asveld@xs4all.nl

Abstract — We consider the following ways of extending a family of languages \( K \) to an “enriched” family \( X(K) \): (i) hyper-algebraic extension \( X = H \) based on iterated parallel substitution, (ii) algebraic extension \( X = A \) obtained by nested iterated substitution, (iii) rational extension \( X = R \) achieved by not self-embedding nested iterated substitution, and (iv) a few subrational extensions \( X = M, C, S, P, F \) based on several kinds of substitution. We introduce full \( X \)-AFL’s, i.e., nontrivial families closed under finite substitution, intersection with regular sets and under \( X \), which turn out to be equivalent to well-known full AFL-structures such as full hyper-AFL \( X = H \), full super-AFL \( X = A \), full substitution-closed AFL \( X = R \), full semi-AFL \( X = S \), etc. Then we establish Canonical Forms for the smallest full \( X \)-AFL \( \hat{X}(K) \) containing \( K \), i.e., we decompose the operator \( \hat{X} \) into simpler operators. Using Canonical Forms for full \( X \)-AFL’s we obtain expressions for the smallest full \( X \)-AFL containing the result of substituting a family of languages into another family.

1 Introduction

In studying closure properties of language families major progress was made since 1969 when Ginsburg, Greibach & Hopcroft [10] introduced the concept of Abstract Family of Languages (AFL) denoting any family closed under the regular operations (union, concatenation and Kleene \(+\)), homomorphism, inverse homomorphism and intersection with regular sets. Using this concept we consider language families as algebras which enables us to deal with closure properties in a more abstract fashion.

As usual in algebra, structures with more restrictive and, on the other hand, more general properties have been considered. Ginsburg & Spanier [11] investigated substitution-closed AFL's generalizing a well-known property of the regular languages, and Greibach

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studied AFL’s closed under nested iterated substitution (super-AFL’s) featuring a property of the context-free languages. In the theory of parallel rewriting (in particular in the case of ETOL languages; cf. [21]) AFL’s closed under iterated parallel substitution (hyper-AFL’s) were intensively investigated [24, 23, 4, 2]. On the other hand many theorems originally proved for AFL’s also hold for weaker structures like semi-AFL (inspired by the linear context-free languages) and trio. For the main results on trio, (semi-)AFL and substitution-closed AFL we refer to a survey by Ginsburg [9].

Van Leeuwen [24] and Salomaa [23] originally introduced (full) hyper-AFL’s and a related rewriting system called \( K \)-iteration grammar, where \( K \) refers to an arbitrary family of languages. A \( K \)-iteration grammar is essentially an ETOL system [21] in which each table (i.e., finite substitution) has been replaced by an arbitrary \( K \)-substitution. The family of languages generated by \( K \)-iteration grammar is called the hyper-algebraic extension of the family \( K \) and will be denoted by \( H(K) \). According to [24, 23, 2] a nontrivial family \( K \) is a full hyper-AFL if and only if (i) \( K \) is closed under finite substitution, (ii) \( K \) is closed under intersection with regular sets, and (iii) \( K \) is hyper-algebraically closed, i.e., \( H(K) = K \). In [2] it was shown that the smallest full hyper-AFL \( \hat{H}(K) \) containing the family \( K \) equals the hyper-algebraic extension of the smallest prequasoid \( \Pi(K) \) (i.e., the smallest nontrivial family satisfying (i) and (ii)) containing \( K \), i.e., \( \hat{H}(K) = H\Pi(K) \). This “Canonical Form Theorem” means that the operator \( \hat{H} \) can be decomposed into a single product (composition) of the simpler operators \( H \) and \( \Pi \).

In this paper we investigate similar ways of connecting other well-known AFL-structures with rewriting systems (like restricted kinds of \( K \)-iteration grammars such as, e.g., context-free \( K \)-grammars introduced in [25]), in such a way that

(A) the corresponding extension \( X \) (i.e., the family of languages generated by that particular kind of restricted iteration grammars) —which may be considered as an operator on families of languages— characterizes the original AFL-structure by means of the conditions: (1) containment of a nontrivial language, (2) closure under finite substitution, (3) closure under intersection with regular sets, and (4) closure under the operator \( X \). A family satisfying (1)–(4) will be called a full \( X \)-AFL. Thus \( K \) is a full \( X \)-AFL when \( \Pi(K) = K \) and \( X(K) = K \).

(B) a Canonical Form Theorem for \( \hat{X}(K) \) (i.e., the smallest full \( X \)-AFL containing \( K \)) holds or, equivalently, it is possible to factorize the operator \( \hat{X} \) into a single product of \( X \) and \( \Pi \): \( \hat{X}(K) = X\Pi(K) \).

The particular types of AFL-structures that will be considered in this paper are full trio, full semi-AFL, full pseudo-AFL (i.e., full semi-AFL closed under concatenation), full Kleene-AFL (i.e., full trio closed under Kleene *), full AFL [9], full substitution-closed AFL [11], full super-AFL [14, 15], full hyper(1)-AFL [24, 4, 8] and full hyper-AFL [24, 23, 4, 2].

From (A) it should be clear that \( \hat{X}(K) = \bigcup\{\Omega(K) \mid \Omega \in \{\Pi, X\}^*\} \), or in a more algebraic notation, that \( \hat{X}(K) = \{\Pi, X\}^*(K) \). Thus the Canonical Forms as mentioned in (B) imply that instead of the infinite set of operators \( \{\Pi, X\}^* \) we only need to consider
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the application of one single operator $X$. Therefore Canonical Forms can be successfully
used in proving certain families to be full $X$-AFLs and in establishing properties of $X(K)$.

All extensions in this paper are obtained either "directly" by natural generalizations
of basic families (like ETOL, EOL, context-free and regular languages) or "indirectly" by
fixing parameters in extensions already defined in a "direct" way. Investigating extensions
of language families instead of basic families enables us to uncover structural features and
arguments for the rewriting mechanisms involved, rather than strictly combinatorial proofs
usually given in dealing with basic families. This rather algebraic approach to language
theory does not only provide a unifying framework for investigating closure properties
(to which the present paper is mainly restricted) but has also successfully been applied in
other areas of formal language theory; cf. e.g., [25, 26, 27]. This approach also emphasizes
again the principal role played by the concept of prequasoid (or, equivalently, the notion
of nondeterministic generalized sequential machine with accepting states; cf. [18], Chapter
9) in dealing with families closed under several kinds of (iterated) substitution [2, 8, 12,
15, 19, 23, 24, 26].

This paper is organized into five sections of which this introductory section is the
first. Section 2 contains preliminaries from formal language theory, definitions of the basic
rewriting systems, the corresponding extensions and full $X$-AFL’s. We conclude Section
2 with proving basic properties of these extensions and with a few examples. In Section
3 we establish Canonical Forms for full hyper-AFL, full hyper(1)-AFL, full algebraic AFL
(i.e., full super-AFL) and full rational AFL (i.e., full substitution-closed AFL). Canonical
Forms for the other cases (i.e., ful trio, full semi-AFL, full pseudo-AFL, full Kleene-AFL,
full AFL) are proved in Section 4. The results of Section 3 and 4 are applied in Section 5 to
the family obtained by substituting languages from a family into languages from another
family. Section 5 also contains a few generalizations of results of Section 4 and a example
of the application of Canonical Forms and related theorems in proving certain families to
be particular kinds of full AFL-structures.

2 Definitions and Basic Properties

For terminology and basic results in formal language theory we refer to [22] or to [18].
Moreover, the reader is assumed to be familiar with standard concepts from the theory of
parallel rewriting [17] and from AFL-theory [9].

Let $\Sigma_\omega$ be an arbitrary but fixed countably infinite set of symbols. A family (of lan-
guages) is a collection of languages over finite subsets, called alphabets, of $\Sigma_\omega$.

Let $\lambda$ denote the empty word. A language $L$ is nontrivial if $L$ is nonempty and $L \neq \{\lambda\}$. A family is called nontrivial when it contains at least one nontrivial language.

Let ONE be the family consisting of all singletons, i.e., ONE = $\{\{w\} \mid w \in \Sigma_\omega^*\}$,
and let SYMBOL be the subfamily containing all singletons of length 1, i.e., SYMBOL = $\{\{\alpha\} \mid \alpha \in \Sigma_\omega\}$. Moreover, we define the families ALPHA and STAR by ALPHA = $\{\{\alpha_1, \ldots, \alpha_n\} \mid n \geq 1, \alpha_1, \ldots, \alpha_n \in \Sigma_\omega\}$ and STAR = $\{\{\alpha^*\} \mid \alpha \in \Sigma_\omega\}$, respectively. We

ETOL- and indexed [1] languages by FIN, REG, CF, OL, EOL, TOL, ETOL and INDEX, respectively.

Let $K$ be a family. A $K$-substitution $\tau$ is a function on an alphabet $V$ such that for each $\alpha \in V$, $\tau(\alpha)$ is a language in $K$. The function $\tau$ is extended in the usual way to words by $\tau(\lambda) = \{\lambda\}$; $\tau(\alpha_1 \cdots \alpha_n) = \tau(\alpha_1) \cdots \tau(\alpha_n)$ with $\alpha_1, \ldots, \alpha_n \in V$ $(n \geq 1)$, and to languages by $\tau(L) = \bigcup \{\tau(w) \mid w \in L\}$ for each $L \subseteq V^*$. A $K$-substitution $\tau : V \to K$ with $\tau(\alpha) \subseteq V^*$ for each $\alpha$ in $V$ is called a $K$-substitution over $V$.

A ONE-substitution is usually called a homomorphism, and in case $K$ equals FIN or REG, $\tau$ is called a finite or regular substitution, respectively.

With $K_1/K_2$ we denote the family obtained by all $K_1$-substitutions applied to $K_2$-languages, i.e., $K_1/K_2 = \{\tau(L) \mid \tau$ is a $K_1$-substitution and $L \in K_2\}$. This binary operation on families is neither commutative, nor associative, i.e., neither $K_1/K_2 = K_2/K_1$, nor $K_1/(K_2/K_3) = (K_1/K_2)/K_3$ holds in general [11]. But the inclusion $K_1/(K_2/K_3) \subseteq (K_1/K_2)/K_3$ does hold [11], whereas equality holds whenever $K_2$ is closed under isomorphism (i.e., renaming of symbols) [11]. In the sequel the operator $/$ will be often applied to families closed under isomorphism and consequently parentheses in expressions will be omitted which results for instance in $K_1/K_2/K_3$.

A family $K_2$ is closed under $K_1$-substitution if $K_1/K_2 \subseteq K_2$. The closure of $K_2$ under $K_1$-substitution is the smallest family containing $K_2$ and closed under $K_1$-substitution. The family $K_1$ is closed under substitution into the family $K_2$ if $K_1/K_2 \subseteq K_1$. The closure of $K_1$ under substitution into $K_2$ is the smallest family containing $K_1$ and closed under substitution into $K_2$. A family is substitution closed when $K/K \subseteq K$. The substitution closure of $K$ is the smallest substitution-closed family $K_\infty$ containing $K$.

Let $[K_1/K_2] = K_2$ and by induction $[K_1/K_2]^n = K_1/(\bigcup_{i=0}^{n}[K_1/K_2]^i)K_2$ $(n \geq 0)$ and let $[K_1/K_2]^{\infty} = \bigcup_{n=0}^{\infty}[K_1/K_2]^n$. Similarly let $K_1/[K_2] = K_1$ and by induction $K_1/[K_2]^n = (\bigcup_{i=0}^{n}K_1/K_2)^i)/K_2$ $(n \geq 0)$ and let $K_1/[K_2]^{\infty} = \bigcup_{n=0}^{\infty}K_1/[K_2]^n$.

Lemma 2.1. [11]

1. $[K_1/K_2]^{\infty}$ is the closure of $K_2$ under $K_1$-substitution.
2. $K_1/[K_2]^{\infty}$ is the closure of $K_1$ under substitution into $K_2$.
4. If $K \subseteq K/K$, then $K_\infty = [K/K]^{\infty}$.

We define the following operators on families of languages: $\Theta(K) = \text{ONE}/K$, $\Phi(K) = \text{FIN}/K$, and $\Delta(K) = \{L \cap R \mid L \in K, \ R \in \text{REG}\}$.

The notion of prequasoid was defined in [2] as a slightly weaker variant of the quasoid introduced by Van Leeuwen [24].

Definition. A family $K$ is a prequasoid if

1. $K$ is nontrivial,
2. $K$ is closed under finite substitution, i.e., $\Phi(K) \subseteq K$,
3. $K$ is closed under intersection with regular languages, i.e., $\Delta(K) \subseteq K$.

A quasoid is a prequasoid containing at least one infinite language.
Each (pre)quasoid contains all regular (finite, respectively) languages. Consequently REG (FIN respectively) is the smallest (pre)quasoid, and FIN is the only prequasoid which is not a quasoid [2]. For the smallest prequasoid \( \Pi(K) \) containing \( K \) we have

**Lemma 2.2.** If \( K \) is a nontrivial family, then \( \Pi(K) = \Theta \Delta \Phi(K) \).

**Proof.** It is well known that a nontrivial family is a prequasoid if and only if it is closed under mappings induced by a-NGSM's (nondeterministic generalized sequential machines with accepting states; cf. [18] for definitions and details). Moreover each a-NGSM mapping \( T \) can be written as \( T(L) = h(f(L) \cap R) \), where \( h \) is a homomorphism, \( R \) is a regular set and \( f \) is a finite substitution [18]. Since a-NGSM mappings are closed under composition [7], we have \( \Pi(K) = \bigcup_{n=0}^{\infty} (\Theta \Delta \Phi)^n(K) = \Theta \Delta \Phi(K) \). □

The relation between substitution and the operator \( \Delta \) is given by the following result established by Ginsburg & Spanier [11].

**Lemma 2.3.** [11]

1. \( \Delta(K_1/K_2) \subseteq \Delta(K_1)/\Delta \Phi(K_2) \),
2. \( \Delta(K_1)/\Delta(K_2) \subseteq \Theta \Delta((\text{FIN} \cup K_1)/\Phi(K_2)) \).

The family \( K \) is closed under iterated (parallel) substitution if for all finite sets \( U \) of \( K \)-substitutions and for all \( L \) in \( K \), \( U^*(L) = \bigcup \{ \tau_1 \cdots \tau_n(L) \mid n \geq 0, \tau_1, \ldots, \tau_n \in U \} \) is in \( K \). The family \( K \) is closed under single iterated (parallel) substitution, when \( U^*(L) \) is in \( K \) for each \( L \) in \( K \) and for each singleton \( U = \{ \tau \} \), i.e., when \( \bigcup \{ \tau^n(L) \mid n \geq 0 \} \) is in \( K \). A \( K \)-substitution \( \tau \) over \( V \) is called nested, if for each \( \alpha \) in \( V \), \( \alpha \in \tau(\alpha) \). If for each \( L \) in a family \( K \) and each nested \( K \)-substitution \( \tau \), \( \bigcup \{ \tau^n(L) \mid n \geq 0 \} \) is in \( K \), then \( K \) is said to be closed under nested iterated substitution.

A full trio is a nontrivial family closed under homomorphism, inverse homomorphism and intersection with regular sets. A full semi-AFL is a full trio closed under union. We call a semi-AFL closed under concatenation a full pseudo-AFL. Thus a full AFL is a full pseudo-AFL closed under Kleene \(*\). We call a full trio closed under Kleene \(*\) a full Kleene-AFL. In [9] it has been shown that each full Kleene-AFL closed under either union or concatenation is a full AFL. Similarly it is straightforward to show that each full trio closed under concatenation is a full pseudo-AFL. A full substitution-closed AFL [full super-AFL, full hyper(1)-AFL, full hyper-AFL] is a full AFL closed under substitution [nested iterated substitution, single iterated substitution, iterated substitution, respectively]. Let \( \hat{\mathcal{M}}, \hat{\mathcal{C}}, \hat{\mathcal{S}}, \hat{\mathcal{P}}, \hat{\mathcal{F}}, \hat{\mathcal{R}}, \hat{\mathcal{A}}, \hat{\mathcal{H}_1} \) and \( \hat{\mathcal{H}} \) denote the least full trio, full Kleene-AFL, full semi-AFL, full pseudo-AFL, full AFL, full substitution-closed AFL, full super-AFL, full hyper(1)-AFL and full hyper-AFL containing \( K \), respectively. In Sections 3 and 4 we will show that all these AFL-structures are different.

Ginsburg & Spanier [11] proved the following

**Lemma 2.4.** If \( K \) is a quasoid, then \( K_\infty \) is a full substitution-closed AFL. □

We now come to the main definitions. They paraphrase and slightly generalize the original definitions introduced by Van Leeuwen [24, 25] and Salomaa [23] (cf. [2]), in order...
to fit standard AFL-structures in the framework of family extensions; cf. Section 3 and 4.

**Definition.** Let $K_1$, $K_2$ and $K_3$ be families. A language $L$ is called *hyper-algebraic* over $(K_1, K_2, K_3)$ when there exist (i) an alphabet $V$, (ii) a terminal alphabet $\Sigma$, (iii) an initial language $L_0 \subseteq V^*$ in $K_1$, (iv) an initial $K_2$-substitution $\tau_0$ over $V$, and (v) a finite set $U$ of $K_3$-substitutions over $V$ such that $L = U^*(\tau_0(L)) \cap \Sigma^*$. If each $\tau$ in $U$ is nested, then $L$ is called *algebraic* over $(K_1, K_2, K_3)$, whereas such an algebraic $L$ is called *rational* over $(K_1, K_2, K_3)$ when $U$ is not self-embedding, i.e., if for all $u$ in $U^*$ and for all $\alpha$ in $V$ the implication

$$w_1\alpha w_2 \in u(\alpha) \Rightarrow (w_1 = \lambda \text{ or } w_2 = \lambda)$$

holds, where $w_1, w_2 \in V^*$. The *hyper-algebraic [algebraic, rational, respectively] extension* $H(K_1, K_2, K_3) [A(K_1, K_2, K_3), R(K_1, K_2, K_3)]$ of $(K_1, K_2, K_3)$ is the family of all languages hyper-algebraic [algebraic, rational] over $(K_1, K_2, K_3)$. \(\square\)

We are mainly interested in several less general extensions which are easily obtained as particular cases.

**Definition.** Let $K$ be a family. The *hyper-algebraic, algebraic and rational extension* of $K$ are respectively

- $H(K) = H(\text{SYMBOL, SYMBOL, } K)$,
- $A(K) = A(\text{SYMBOL, SYMBOL, } K)$, and
- $R(K) = R(\text{SYMBOL, SYMBOL, } K)$. \(\square\)

Let $K_1 = K_2 = \text{SYMBOL}$ and $K_3 = K$. Let $V$, $\Sigma$, $U$, $\tau_0$ and $L_0$ be as in the previous definition. The construct $G = \langle V, \Sigma, U, S \rangle$, where $\{S\} = \tau_0(L_0)$ is called a *$K$-iteration grammar*, if $U$ is a finite set of $K$-substitutions over $V$. We call $G$ a *context-free $K$-grammar* [regular $K$-grammar] if each $\tau$ in $U$ is a nested $K$-substitution over $V$ [and $U$ is not self-embedding]. The grammar $G$ generates a language $L(G) = U^*(S) \cap \Sigma^*$ which is a member of $H(K)$ [$A(K)$, $R(K)$, respectively].

The $m$-restricted extensions $H_m(K)$, $A_m(K)$ and $R_m(K)$ are the subfamilies of $H(K)$, $A(K)$ and $R(K)$ respectively, generated by grammars containing at most $m$ $K$-substitutions ($m \geq 1$).

Another type of less general extensions are the following, which we call the *subrational extensions* of $K$:

- $M(K) = R(\text{SYMBOL, } K, \text{FIN})$,
- $C(K) = R(\text{STAR, } K, \text{FIN})$,
- $S(K) = R(\text{ALPHA, } K, \text{FIN})$,
- $P(K) = R(\text{FIN, } K, \text{FIN})$,
- $F(K) = R(\text{REG, } K, \text{FIN})$.

We call a family $K$ *$\sigma$-simple* when (i) $K$ contains a SYMBOL-language, and (ii) $K$ is closed under isomorphism (“remaining of symbols”). A $\sigma$-simple family closed under union with SYMBOL-languages is called *$\alpha$-simple*. Clearly, SYMBOL [ALPHA, respectively] is the smallest $\sigma$-simple [$\alpha$-simple] family and it is contained in each $\sigma$-simple [$\alpha$-simple]
family. Obviously, each prequasoid is $\sigma$-simple. Consider a nontrivial language $L \subseteq \Sigma^*$ from a prequasoid $K$. Let $f$ be a finite substitution and $h$ a homomorphism defined by $f(a) = \{a, b, e\}$ for each $a \in \Sigma$, $e \notin \Sigma$ and $h(e) = \lambda$, $h(\alpha) = \alpha$ for $\alpha \in \Sigma \cup \{b\}$. Then $L \cup \{b\} = h(f(L) \cap (\Sigma^* \cup be^*))$ is in $K$. Hence each prequasoid is $\alpha$-simple.

Our first result deals with the relation between extensions and the corresponding $m$-restricted variants.

**Theorem 2.5.**

1. If $K$ is $\sigma$-simple, then $H_2(K) = H_m(K) = H(K)$ for each $m \geq 2$.
2. If $K$ is $\alpha$-simple, then $A_1(K) = A_m(K) = A(K)$ and $R_1(K) = R_m(K) = R(K)$ for each $m \geq 1$.

**Proof.** (1) was already established in [2]. Thus it remains to show that for each $m \geq 1$, $A(K) \subseteq A_m(K) \subseteq A_1(K)$ and similarly for $R$, since the reverse inclusions are obvious.

Let $G = (V, \Sigma, U, S)$ be a context-free $K$-grammar with $U = \{\tau_1, \ldots, \tau_m\}$ for some $m \geq 2$. Define for each $k$ with $1 \leq k \leq m$ an isomorphism $\varphi_k(\alpha) = \alpha_k$ ($\alpha$ in $V$; all $\alpha_k$’s are new symbols) and extend these isomorphisms in the usual way to words and languages. Define a new alphabet $V_0 = V \cup \{\varphi_k(\alpha) \mid \alpha \in V, 1 \leq k \leq m\}$. Consider the context-free $K$-grammar $G_0 = (V_0, \Sigma, \tau, S)$ where the nested $K$-substitution $\tau$ over $V_0$ is defined by

$$
\tau(\alpha) = \{\alpha, \alpha_1\} \quad \alpha \in V, \\
\tau(\alpha_k) = \{\alpha_k\} \cup C_k \cup \tau_k(\alpha) \quad \text{with} \quad C_k = \emptyset \quad \text{iff} \quad k = m, \\
\quad \text{and} \quad C_k = \{\alpha_{k+1}\} \quad \text{iff} \quad l \leq k \leq m-1.
$$

The basic idea of the simulation of $G$ by $G_0$ is the following: each occurrence of each symbol $\beta$ in $V_0$ may be object to the following replacements (in an “asynchronous way”): (i) changing into $\alpha_{k+1}$ (if $\beta = \alpha_k$, $1 \leq k \leq m-1$) or into $\alpha_1$ (if $\beta = \alpha$), i.e., from $\alpha$ we can reach $\alpha_j$ for each $j$ ($1 \leq j \leq m$), (ii) substituting $\tau_j(\alpha)$ into that particular instance of $\beta = \alpha_j$, i.e., we simulate the application of $\tau_j$ on that occurrence of $\alpha_j$ while the subscript $j$ is removed.

By this construction we obtain $L(G_0) = L(G)$ and hence $A(K) \subseteq A_m(K) \subseteq A_1(K)$ for each $m \geq 1$. Since the construction preserves the not self-embedding property of the grammar, a similar conclusion holds in the rational case.

Note that 2.5(2) is obvious, whenever $K$ is closed under union. In the hyper-algebraic case 2.5(1) is the best possible result, i.e., in general one cannot reduce the number of substitutions in a $K$-iteration grammar to one [21, 8].

By relating subrational extensions to (finitely repeated) substitutions we obtain a considerable simplification.

**Theorem 2.6.** $\mathcal{R}(K_1, K, \text{FIN}) = \text{REG}/(K/K_1)$, and consequently

$$
M(K) = \text{REG}/K, \\
C(K) = \text{REG}/(K/\text{STAR}), \\
S(K) = \text{REG}/(K/\text{ALPHA}),
$$
\[ P(K) = \text{REG}/(K/\text{FIN}), \text{ and} \]
\[ F(K) = \text{REG}/(K/\text{REG}), \]

whereas the parentheses may be omitted whenever \( K \) is closed under isomorphism.

**Proof.** Suppose \( L \in \text{REG}/(K/K_1) \) and let \( L = \rho \tau_0(L_0) \) where \( L \subseteq \Delta^* \) is in \( K_1 \), \( \tau_0 : \Delta \rightarrow K \) is a \( K \)-substitution with \( \tau(\alpha) \subseteq B^* \) for each \( \alpha \) in \( \Delta \), and \( \rho : B \rightarrow \text{REG} \) is a regular substitution with \( \rho(\beta) \subseteq \Sigma_\beta^* (\beta \in B) \), and \( \Sigma = \bigcup \{ \Sigma_\beta \mid \beta \in B \} \). For each \( \beta \in B \), let \((V_\beta, \Sigma_\beta, P_\beta, \beta)\) be an ordinary regular grammar generating \( \rho(\beta) \) such that all nonterminal alphabets \( V_\beta - \Sigma_\beta \) are mutually disjoint. Let \( U = \{ \tau \} \) where \( \tau \) is a substitution over \( \bigcup \{ V_\beta \mid \beta \in B \} \cup \Delta \) defined by
\[
\tau(\gamma) = \{ \gamma \} \cup \{ w \mid (\gamma, w) \in P_\beta \} \in \text{FIN} \quad \text{iff} \quad \gamma \in V_\beta - \Sigma_\beta \quad (\beta \in B),
\]
\[
\tau(\gamma) = \{ \gamma \} \quad \text{otherwise.}
\]

Clearly, \( U \) is not self-embedding and therefore we have \( L = \rho \tau_0(L_0) = U^*(\tau_0(L_0)) \cap \Sigma^* \in R(K_1, K, \text{FIN}) \).

Conversely, let \( L \in R(K_1, K, \text{FIN}) \), i.e., there exist alphabets \( V \) and \( \Sigma \), a language \( L_0 \) in \( K_1 \), a \( K \)-substitution \( \tau_0 \) over \( V \), and a finite set \( U \) of finite substitutions over \( V \), such that \( L = U^*(\tau_0(L_0)) \cap \Sigma^* \). Since \( \text{FIN} \) satisfies the conditions required in 2.5(2) we can (by a similar argument as in the proof of 2.5(2)) restrict our attention to the case \( U = \{ \tau \} \). Consider for each \( \alpha \) in \( V \) the grammar \( G_\alpha = (V, \Sigma, \tau, \alpha) \). Since each \( G_\alpha \) is not self-embedding and \( \tau \) is a finite substitution, \( L(G) = (\bigcup \{ \tau^n(\alpha) \mid n \geq 0 \} \cap \Sigma^* \) is regular (cf., e.g., [22]). Hence there exists a regular substitution \( \rho : V \rightarrow \text{REG} \) with \( \rho(\alpha) = L(G_\alpha) \) \( (\alpha \in V) \) and consequently \( L = \rho \tau_0(L_0) \in \text{REG}/(K/K_1) \). \( \square \)

**Definition.** Let \( X \) be a symbol from \( \{ H, H_1, A, R, F, P, S, C, M \} \). A family \( K \) is a full \( X \)-AFL if
\[(i) \quad K \text{ is a prequasoid, i.e., } \Pi(K) = K, \text{ and} \]
\[(ii) \quad K \text{ is equal to its } X \text{-extension, i.e., } X(K) = K. \square \]

In the subsequent two sections we relate full \( X \)-AFL’s to well-known AFL-structures.

**Examples.**
1. \( H(\text{ONE}) = \text{EDTOL}; \ H(\text{FIN}) = H(\text{REG}) = \text{ETOL}; \ H(\text{INDEX}) = \text{INDEX} \), hence ETOL and INDEX are full \( H \)-AFL’s [24, 23, 4].
2. \( H_1(\text{ONE}) = \text{EDOL}; \ H_1(\text{FIN}) = \text{EOL} [24]. \)
3. \( A(\text{FIN}) = A(\text{REG}) = A(\text{CF}) = \text{CF} [25, 26, 14, 15] \), hence CF is a full \( A \)-AFL.
4. \( R(\text{FIN}) = R(\text{REG}) = \text{REG} \) (cf. 2.6 and 3.3 below), hence REG is a full \( R \)-AFL.
5. For each \( X \) from \( \{ M, C, S, P, F \} \), we have \( X(\text{FIN}) = X(\text{REG}) = \text{REG} \) (cf. 2.6) and consequently REG is a full \( X \)-AFL. \( \square \)

### 3 Canonical Forms for Full \( H \)-, \( H_1 \)-, \( A \)- and \( R \)-AFL’s
In [2] it was already shown that a family is a full \( H \)-AFL if and only if it is a full hyper-AFL (cf. Section 1, [24, 23]). By a similar argument we obtain that a family is a full \( H_1 \)-AFL
Lemma 3.1. If $K$ is $\alpha$-simple, then $K_\infty \subseteq R(K)$.

Proof. Since $\text{SYMBOL} \subseteq K$ we have $K \subseteq K/K$ and hence by 2.1(4), $K_\infty = [K/]^*K$. Moreover $K \subseteq K/K$ also implies $[K/]^{n+1}K = K/(\bigcup_{i=0}^{n}[K/]iK) = K/[K/]^{n}K$.

In order to establish the inclusion $K_\infty \subseteq R(K)$ we show that for all natural numbers $p$, $[K/]pK \subseteq R(K)$. The proof is by induction on $p$.

Initial step ($p = 0$): $[K/]0K = K$. Let $L_0 \subseteq \Sigma^*$ be in $K$ and consider the context-free $K$-grammar $G = (V, \Sigma, \{\tau\}, S)$ where $V = \Sigma \cup \{S\}$, $S \notin \Sigma$, $\tau(S) = \{S\} \cup L_0$ and $\tau(a) = \{a\}$ for each $a$ in $\Sigma$. Clearly, $G$ is not self-embedding and $L(G) = L_0$ is in $R(K)$.

Induction step: Suppose $[K/]pK \subseteq R(K)$, then we have to show that $[K/]^{p+1}K \subseteq R(K)$. By the induction hypothesis we obtain $[K/]^{p+1}K = K/[K/]pK \subseteq K/R(K)$. Thus it only remains to prove that $K/R(K) \subseteq R(K)$.

Let $L' \subseteq \Sigma^*$ be a language in $R(K)$ generated by the regular $K$-grammar $G = (V, \Sigma, \{\tau\}, S)$, and let $g$ be a $K$-substitution on $\Sigma$ with $\bigcup\{g(\alpha) \mid \alpha \in \Sigma\} \subseteq \Sigma_0$ for some alphabet $\Sigma_0$. Without loss of generality we may assume that $V \cap \Sigma_0 = \emptyset$. Consider the context-free $K$-grammar $G_0 = (V_0, \Sigma_0, U_0, S)$ where $V_0 = V \cup \Sigma_0$, $U_0 = \{\tau_0, \tau_1\}$ with

$$
\tau_0(\alpha) = \{\alpha\}, \quad \text{for } \alpha \text{ in } V_0 - \Sigma,
$$

$$
\tau_0(\alpha) = \{\alpha\} \cup g(\alpha), \quad \text{for } \alpha \text{ in } \Sigma,
$$

$$
\tau_1(\alpha) = \tau(\alpha), \quad \text{for } \alpha \text{ in } V,
$$

$$
\tau_1(\alpha) = \{\alpha\}, \quad \text{for } \alpha \text{ in } \Sigma_0.
$$

Then clearly, $G_0$ is a not self-embedding context-free $K$-grammar and $L(G_0) = g(L(G)) = g(L') \subseteq R(K)$.

This completes the induction and establishes the inclusion $K_\infty \subseteq R(K)$. \hfill \Box

Lemma 3.2. If $K$ is a prequasoid, then $R(K) \subseteq K_\infty \cup \text{REG}$.

Proof. If $K$ is a prequasoid but not a quasoid, then $K = \text{FIN}$ and hence $R(\text{FIN}) = \text{REG} \subseteq K_\infty \cup \text{REG}$.

Let $K$ be a quasoid, then $\text{REG} \subseteq K \subseteq K_\infty$. By 2.5(2) we only have to prove that $R_1(K) \subseteq K_\infty$. We first show that $R_1(K_\infty) \subseteq K_\infty$. The proof of this inclusion is a generalization of the argument that each non-self-embedding context-free grammar generates a regular language; cf. [22].

Consider a 1-restricted regular $K_\infty$-grammar $G = (V, \Sigma, \{\tau\}, S)$. Without loss of generality we may assume that $\tau(a) = \{a\}$ for each $a$ in $\Sigma$. (Otherwise, we introduce for each $a$ in $\Sigma$ a new nonterminal symbol $A_a$ and we define an isomorphism $\varphi(a) = A_a$ ($a \in \Sigma$), $\varphi(\alpha) = \alpha$ ($\alpha \in V - \Sigma$). We replace $\tau$ by $\tau_0$ with $\tau_0(\alpha) = \varphi(\tau(\alpha))$ iff $\alpha \in V - \Sigma$, $\tau_0(\alpha) = \{a\}$ iff $a \in \Sigma$, $\tau_0(A_a) = \{A_a, a\} \cup \varphi(\tau(a))$ for each $a$ in $\Sigma$.)
Moreover we assume that for each \( \alpha \) in \( V \) there is a sequence \( u \) in \( \tau^* \) such that there is a word in \( u(S) \) that contains an occurrence of \( \alpha \). Otherwise we can remove \( \alpha \) from \( V \) and intersect all languages involved in \( G \) with \( (V - \{ \alpha \})^* \) (due to the fact that \( K_\infty \) is a full AFL by 2.4), without affecting \( L(G) \). We consider the following cases:

\textbf{Case 1:} For each \( A \) in \( V - \Sigma \), there is a sequence \( u \) in \( \tau^* \) such that \( u(A) \) contains a word in which \( \tau \) occurs.

If \( w \in \tau(A) \) is an arbitrary word containing a nonterminal symbol, then it is of one of the four forms: (i) \( w = \varphi B \psi \), (ii) \( w = \varphi B \), (iii) \( w = B \psi \), or (iv) \( w = B \), where \( \varphi \) and \( \psi \) are nonempty words over \( V \), and \( B \in V - \Sigma \). If \( w \) satisfies (i) we must have by the assumption of Case 1: there exist sequences \( u_1 \) and \( u_2 \) in \( \tau^* \) such that

\[ u_1 u_2 \tau(A) \supseteq u_1 (\varphi B \psi) \supseteq u_1 (\varphi \varphi_1 S \psi \psi_1) \supseteq \{ \varphi \varphi_1 \varphi_2 A \psi \psi_1 \psi \} \]

for some (possibly empty) words \( \varphi_1 \), \( \varphi_2 \), \( \psi_1 \) and \( \psi_2 \) over \( V \). But then, since \( \varphi \) and \( \psi \) are nonempty, \( G \) is a self-embedding context-free \( K_\infty \)-grammar which contradicts the assumption that \( G \) is regular. We obtain the same contradiction if \( \tau(A) \) contains words of both forms (ii) and (iii). Thus if \( \tau(A) \) contains a word of the form (ii), then all words of the form (ii) contained in \( \tau(A) \) the word \( \varphi \) is in \( \Sigma^* \). (Otherwise \( \tau(A) \) would also contain a word of the form (i) or (iii).) Hence \( G \) is a “right-linear” context-free \( K_\infty \)-grammar, i.e., for each \( A \in V - \Sigma \), we have \( \tau(A) \subseteq \Sigma^*(V - \Sigma) \cup \Sigma^* \). By a similar argument we conclude that if \( \tau(A) \) contains a word of the form (iii), then \( G \) is “left linear”, i.e., for each \( A \) in \( V - \Sigma \), \( \tau(A) \subseteq (V - \Sigma) \Sigma^* \cup \Sigma^* \).

So if \( G \) is right linear, then we have \( \tau(A) = \bigcup \{ L_{AX} X \} \ | \ X \in V - \Sigma \} \cup L_A \) for each \( A \) in \( V - \Sigma \), where \( L_A \) and \( L_{AX} \) are languages over \( \Sigma \). (The left-linear case is similar.) Note that \( L_{AX} \) is empty whenever \( X \) does not occur in \( \tau(A) \). Since \( K_\infty \) is a full AFL (by 2.4), we have that \( L_A \), and each (nonempty) \( L_{AX} \) are in \( K_\infty \) and, consequently, \( R_1(K_\infty) \subseteq K_\infty/\text{REG} = K_\infty \); cf. 2.4 and [9, 11].

\textbf{Case 2:} There exists a nonterminal symbol \( A \) such that for no words \( \varphi \) and \( \psi \) in \( V^* \) and for no sequence \( u \) in \( \{ \tau \}^* \), \( \varphi S \psi \in u(A) \).

The proof of \( L(G) \) being in \( K_\infty \) proceeds by induction on the number of nonterminal symbols \( m \). For \( m = 1 \), there is nothing to prove because \( \lambda(S) = \{ S \} \).

Assume that the assertion holds for \( m = n \). Let the number of nonterminal symbols in \( V - \Sigma \) be \( n + 1 \). Consider the context-free \( K_\infty \)-grammar \( G_1 = (V - \{ S \}, \Sigma, \tau_1, A) \) with

\[ \tau_1(\alpha) = \tau(\alpha) \cap (V - \{ S \})* \quad \alpha \neq S \]

and the context-free \( K_\infty \)-grammar \( G_2 = (V, \Sigma \cup \{ A \}, \tau_2, S) \) with

\[ \tau_2(A) = \{ A \}, \quad \tau_2(\alpha) = \tau(\alpha) \quad \alpha \neq A. \]

Then both \( G_1 \) and \( G_2 \) are not self-embedding grammars having \( n \) nonterminal symbols. Now both of the languages \( L(G_1) \) and \( L(G_2) \) are in \( K_\infty \); either by the induction hypothesis or by Case 1. But \( L(G) \) is the result of substituting \( L(G_1) \) for \( A \) in \( L(G_2) \). The fact that \( K_\infty \) is substitution closed implies that \( L(G) \) is in \( K_\infty \). This completes the induction and establishes the inclusion \( R_1(K_\infty) \subseteq K_\infty \).

Finally, together with \( K \subseteq K_\infty \), this yields \( R_1(K) \subseteq R_1(K_\infty) \subseteq K_\infty \). \( \square \)
Theorem 3.3.

(1) If $K$ is a prequasoid, then $R(K) = K_\infty \cup \text{REG}$.

(2) $K$ is a full $R$-AFL if and only if $K$ is a full substitution-closed AFL.

Proof. (1) Lemmas 3.1 and 3.2.

(2) Let $K$ be a full $R$-AFL. Then $\text{FIN} \subseteq K$ and consequently $\text{REG} \subseteq R(K) = K$. Now 3.3(1) implies $R(K) = K = K_\infty$. Hence by 2.4 $K$ is a full substitution-closed AFL.

Conversely, let $K$ be a full substitution-closed AFL. Obviously, $K$ is a prequasoid and by (1): $K = K_\infty = R(K)$, i.e., $K$ is a full $R$-AFL. \hfill \Box

Let $H^0_1(K) = K$, $H^{n+1}_1(K) = H_1H^n_1(K)$ for $n \geq 0$ and $H^*_1(K) = \bigcup_{n=0}^\infty H^n_1(K)$. In the remaining part of this section $X$ will denote a symbol from $\{H, H^*_1, A, R\}$. Let $\overline{X}(K)$ be the least corresponding full $X$-AFL containing $K$ (i.e., $\overline{X}$ equals $\overline{H}$, $\overline{H}_1$, $\overline{A}$ and $\overline{R}$, respectively).

We now prove the main result of this section, i.e., we will decompose the operator $\hat{X}$ into simpler operators like $X$, $\Pi$, $\Delta$ and $\Phi$. Some other factorizations were already known. E.g., Greibach [14] established a result that may be interpreted as $\hat{A}(K) = \Delta A \overline{F}(K)$. Similarly, from 2.4 due to Ginsburg & Spanier [11] we can infer that $\overline{R}(K)$ equals the substitution closure of $\Pi(K)$ provided $K$ contains an infinite language.

Theorem 3.4. (Canonical Form Theorem for Full $H^*$, $H^+_1$, $A$- and $R$-AFL’s)

(1) $\hat{X}(K) = X\Pi(K)$.

(2) If $K$ is nontrivial, then $\hat{X}(K) = X\Delta\Phi(K)$.

Proof. (1) We distinguish the following cases:

$X = H$: In [2] we proved that $H(K)$ is a full $H$-AFL, provided $K$ is a prequasoid, Hence $\overline{H}(K) \subseteq H\Pi(K)$. Conversely, we have $K \subseteq \overline{H}(K)$ which implies $H\Pi(K) \subseteq H\Pi\overline{H}(K) = \overline{H}(K)$.

$X = H^*_1$: Similar to the previous case it suffices to show that $H^*_1(K)$ is a full $H_1$-AFL, whenever $K$ is a prequasoid. So let $K$ be a prequasoid. Obviously, $H^*_1(K)$ is $H_1$-closed and thus it remains to prove that $H^*_1(K)$ is a prequasoid. This will be done by induction. Since $\text{FIN} \subseteq K$, we have $H_1(\text{EOL}) \subseteq H^2_1(K)$. Applying a result from [23] yields that $H^2_1(K)$ is a full AFL. Suppose $H^n_1(K)$, $n \geq 2$ is a full AFL. Then $H^{n+1}_1(K) = H_1H^n_1(K)$ is also a full AFL [23], which completes the induction.

$X = A$: Let $K$ be a prequasoid. We will show that $A(K)$ is a full $A$-AFL. Since the proof that $A(K)$ is a prequasoid is rather standard (cf. [26, 2]) and the inclusion $A(K) \subseteq AA(K)$ is trivial, it remains to establish that $AA(K) \subseteq A(K)$.

Let $G = (V, \Sigma, \tau, S)$ be a context-free $A(K)$-grammar, i.e., $\tau$ is a nested $A(K)$-substitution. For each $\alpha$ in $V$, let $G_\alpha = (V_\alpha, V, \tau_\alpha, S_\alpha)$ be a context-free $K$-grammar (i.e., each $\tau_\alpha$ is a nested $K$-substitution) such that $L(G_\alpha) = \tau(\alpha)$. Obviously, we may assume that all nonterminal alphabets $V_\alpha - V$ are mutually disjoint. Thus we have to show that $L(G) \in A(K)$.

We modify each $G_\alpha$ in such a way that $\tau_\alpha(\beta) = \{\beta\}$ for each $\beta$ in $V$ (cf. the proof of
3.2). Consider the context-free $K$-grammar $G_0 = (V_0, \Sigma, U_0, S)$ where $V_0 = \bigcup \{V_\alpha \mid \alpha \in V\}$ and $U_0 = \{\sigma_\alpha \mid \alpha \in V\}$ with for each $\alpha$ in $V$:

$$\sigma_\alpha(\beta) = \tau_\alpha(\beta)$$

$$\beta \in V_\alpha - V,$$

$$\sigma_\alpha(\beta) = \{\beta, S_\beta\}$$

$$\beta \in V,$$

$$\sigma_\alpha(\beta) = \{\beta\}$$

$$\beta \in V_0 - V_\alpha.$$  

Clearly, $L(G_0) = L(G)$ and hence $L(G)$ is in $A(K)$.

$X = R$: The fact that $R(K)$ is a full $R$-AFL whenever $K$ is a prequasoid directly follows from 3.3 and 2.4.

(2) By (1) and 2.2, we have $\tilde{H}(K) = H\Theta\Delta\Phi(K)$. Therefore it suffices to show that $H\Theta\Delta\Phi(K) \subseteq H\Delta\Phi(K)$.

Let $L \subseteq \Omega^*$ be a nontrivial language in the family $K$. Define a finite substitution $f$ by $f(\alpha) = \{\lambda, \omega\}$ for each $\alpha$ in $\Omega$, where $\omega$ is an arbitrary but fixed word. Then clearly $\{\omega\} = f(L) \cap \{\omega\}$ is in $\Delta\Phi(K)$, i.e., ONE $\subseteq \Delta\Phi(K)$, which enables us to replace homomorphisms (according to $\Theta$) by $\Delta\Phi(K)$-substitutions as follows. Let $G = (V, \Sigma, U, S)$ be a $\Theta\Delta\Phi(K)$-iteration grammar and consider the $\Delta\Phi(K)$-iteration grammar $G_0 = (V_0, \Sigma, U_0, S)$ where $V_0 = \bigcup \{\varphi_\alpha(V) \mid \alpha \in V; \tau \in U\}$ with each $\varphi_\alpha$ is an isomorphism such that all alphabets in this union are mutually disjoint. For each $\tau$ in $U$, we define a substitution $\tau'$ in $U_0$ by $\tau'(\alpha) = L_\alpha$ if and only if $\alpha \in V$, $L_\alpha \subseteq \varphi_\alpha(V)$ is in $\Delta\Phi(K)$ (Note that $\Delta\Phi(K)$ is closed under isomorphism.) and $h_\alpha(L_\alpha) = \tau(\alpha)$ where $h_\alpha$ is the relative homomorphism according to $\Theta$:

$$\tau'(\alpha) = h_\alpha(\alpha)$$

$$\alpha \in \varphi_\alpha(V),$$

$$\tau'(\beta) = \{\beta\}$$

$$\beta \in V_0 - V - \varphi_\alpha(V).$$

By this construction we have $L(G_0) = L(G)$ and consequently $\tilde{H}(K) = H\Delta\Phi(K)$ for each nontrivial $K$.

Since this construction preserves the number of substitutions and the not self-embedding property, whereas nesting can also be incorporated, the same conclusion holds in the other cases.  

As each prequasoid contains FIN, and FIN is the smallest prequasoid we have

**Corollary 3.5.** Let $X$ be a symbol from \{$H, H_1^*, A, R$\}. Then

1. each full $X$-AFL contains the family $X(FIN)$;
2. $X(FIN)$ is the smallest full $X$-AFL.

This corollary implies that $H(FIN) = ETOL$ is the smallest full $H$-AFL [4], $A(FIN) = CF$ is the smallest full $A$-AFL [14], and $R(FIN) = REG$ is the smallest full $R$-AFL [11].

An improvement in the $H_1$-case to $\tilde{H}_1(K) = H_1\Pi(K)$ instead of $H_1^3\Pi(K)$ is impossible because $H_1(FIN) = EOL$ is not even a full AFL [16]. Starting with quasoids rather than prequasoids indeed yields full AFL’s [23] but no full $H_1$-AFL’s or even full $R$-AFL’s [4]. Recently Engelfriet [8] showed that the iteration of $H_1$ (applied to $\Pi(K)$) cannot be reduced to a finite power since $H_1^3(FIN)$ (i.e., the smallest full $H_1$-AFL) gives rise to an infinite
hierarchy of full AFL’s:
\[ H_1^*(\text{FIN}) \subset H_2^*(\text{FIN}) \subset \cdots \subset H_n^*(\text{FIN}) \subset \cdots. \]

For \( X \) equal to \( H, H_1, A \) and \( R \) we denote the class of all full \( X \)-AFL’s over \( \Sigma_\omega \) by \( X\text{-AFL} \).

**Corollary 3.6.** \( R\text{-AFL} \supset A\text{-AFL} \supset H_1\text{-AFL} \supset H\text{-AFL} \).

**Proof.** From the definitions in 2 it is clear that \( R\text{-AFL} \supseteq A\text{-AFL} \supseteq H_1\text{-AFL} \supseteq H\text{-AFL} \).

Since the smallest full \( X \)-AFL’s for \( X \in \{ R, A, H_1, H \} \) are mutually different (i.e., \( \text{REG} \subset\subset \text{CF} \subset\subset H_1^*(\text{FIN}) \subset\subset \text{ETOL} \) (cf. 3.5 and [21, 8]) the inclusions are also proper. \( \square \)

### 4 Canonical Forms for Subrationally Closed Full AFL’s

Let \( X \) be a symbol from \( \{M, C, S, P, F\} \). First we relate full \( X \)-AFL’s to well-known AFL-structures.

In this section we only give detailed proofs for a few typical cases. The other arguments are obtained by straightforward modifications and are left as simple exercises.

**Theorem 4.1.**

1. \( K \) is a full \( M \)-AFL if, and only if, \( K \) is a full trio.
2. \( K \) is a full \( C \)-AFL if, and only if, \( K \) is a full Kleene-AFL.
3. \( K \) is a full \( S \)-AFL if, and only if, \( K \) is a full semi-AFL.
4. \( K \) is a full \( P \)-AFL if, and only if, \( K \) is a full pseudo-AFL.
5. \( K \) is a full \( F \)-AFL if, and only if, \( K \) is a full AFL.

**Proof.** (5) Let \( K \) be a full \( F \)-AFL, i.e., \( K \) is a prequasoid and \( K = R(\text{REG}, K, \text{FIN}) \).

By 2.6 we have \( K = \text{REG}/K/\text{REG} \). This implies (i) \( K \) is closed under \( \Delta \), (ii) \( \text{REG} \subseteq K \), because \( \text{SYMBOL} \subseteq K \), (iii) \( \text{REG}/K \subseteq K \) and \( K/\text{REG} \subseteq K \), since \( \text{SYMBOL} \subseteq \text{REG} \), which means that \( K \) is a full AFL [11].

Conversely, let \( K \) be a full AFL. Clearly, \( K \) is a prequasoid. Moreover, \( K/\text{REG} \subseteq K \) and \( \text{REG}/K \subseteq K \) hold [11]. Applying 2.6 yields \( R(\text{REG}, K, \text{FIN}) = \text{REG}/K/\text{REG} \subseteq K \), whereas the opposite inclusion is obvious. \( \square \)

Let \( \hat{X}(K) \) denote the least full \( X \)-AFL containing \( K \), i.e., if \( X \) equals \( M, C, S, P \) or \( F \), then \( \hat{X} \) is equal to \( \hat{M}, \hat{C}, \hat{S}, \hat{P} \) or \( \hat{F} \), respectively. With respect to Canonical Forms the case \( X = C \) differs from the other cases and will be treated separately; cf. 4.4.

**Theorem 4.2.** (First Canonical Form Theorem for Full \( M-, S-, P- \) and \( F \)-AFL’s.)

Let \( X \) be a symbol from \( \{M, S, P, F\} \) and let \( K \) be a family of languages. Then

1. \( \hat{X}(K) = X\Pi(K) \).
2. If \( K \) is nontrivial, then \( \hat{X}(K) = X\Delta\Phi(K) \).

**Proof.** (1) \( \underline{X = S} \): Similar as in the proof of 3.4 it suffices to show that \( S(K) \) is a
full $S$-AFL provided $K$ is a prequasoid. So let $K$ be a prequasoid, then by 2.6 $S(K) = \text{REG}/K/\text{ALPHA}$ and hence

(i) $\Phi S(K) = \text{FIN}/\text{REG}/K/\text{ALPHA} = \text{REG}/K/\text{ALPHA} = S(K)$;

(ii) $\Delta S(K) = \Delta(\text{REG}/K/\text{ALPHA}) \subseteq \Delta(\text{REG})/\Delta\Phi(K/\text{ALPHA}) = \text{REG}/\Delta(K/\text{ALPHA})$

$= \text{REG}/\{(\bigcup L_i \cap R \mid L_i \in K; R \in \text{REG}\} = \text{REG}/\{(\bigcup (L_i \cap R) \mid L_i \in K; R \in \text{REG}\} = \text{REG}/K/\text{ALPHA} = S(K)$ (The inclusion is obtained by 2.3(1); $i$ goes through any finite index set);

(iii) $SS(K) = \text{REG}/\text{REG}/K/\text{ALPHA}/\text{ALPHA} = \text{REG}/K/\text{ALPHA} = S(K)$.

(i), (ii) and (iii) imply that $S(K)$ is a full $S$-AFL.

$X = F$: Apart from taking REG instead of ALPHA, the cases (i) and (iii) are the same as above. So it remains to show that $\Delta F(K) \subseteq F(K)$. Applying 2.3(1) twice yields: $\Delta F(K) = \Delta(\text{REG}/K/\text{REG}) \subseteq \Delta(\text{REG})/\Delta\Phi(K/\text{REG}) = \text{REG}/\Delta(K/\text{REG}) \subseteq \text{REG}/\Delta(K)/\Delta\Phi(\text{REG}) = \text{REG}/K/\text{REG} = F(K)$.

(2) Let $K_X$ be SYMBOL, ALPHA, FIN and REG for $X$ equal to $M$, $S$, $P$ and $F$, respectively. By (1), 2.2 and the fact that $\Delta\Phi(K)$ is closed under isomorphism we obtain $\hat{\chi}(K) = XV(\hat{\chi}(K)) = X\Theta\Delta\Phi(K) = \text{REG}/\text{ONE}/\Delta\Phi(K)/K_X = \text{REG}/\Delta\Phi(K)/K_X = X\Delta\Phi(K)$. □

From 2.6 and 4.2(1) we may infer other Canonical Forms like $\hat{\chi}(K) = \hat{M}(K)/\text{REG}$ or $\hat{S}(K) = \hat{M}(K)/\text{ALPHA}$ which were originally established in [10] and [19] respectively; cf. [12, 20].

Theorem 4.3. (Second Canonical Form Theorem for Full $M$-, $S$-, $P$- and $F$-AFL’s.)

If $K$ is $\sigma$-simple, then $\hat{\chi}(K) = \Pi X(K) = \Theta\Delta X(K)$ for each $X$ from $\{M, S, P, F\}$.

Proof. The inclusion $K \subseteq \Pi(K)$ and 4.2 imply $X(K) \subseteq X\Pi(K) = \hat{\chi}(K)$ and hence $\Theta\Delta X(K) \subseteq \Pi X(K) \subseteq \Pi \hat{\chi}(K) = \hat{\chi}(K)$. Thus it remains to show that $\hat{\chi}(K) \subseteq \Theta\Delta X(K)$. Distinguish the following cases.

$X = M$: According to 4.2, 2.6 and 2.3(2) we have

$\hat{M}(K) = \text{REG}/\Delta\Phi(K) \subseteq \Theta\Delta((\text{REG} \cup \text{FIN})/\Phi\Phi(K)) = \Theta\Delta M(K)$.

$X = S$: By 4.2 and the previous case we have $\hat{S}(K) = \{\bigcup L_i \mid L_i \in \hat{M}(K)\}; 1 \leq i \leq n\}$ = $\{\bigcup L_i \mid L_i \in \Theta\Delta M(K); 1 \leq i \leq n\}$, i.e., $L_i = h_i(L_i' \cap R_i)$ with $L_i' \in M(K)$, $R_i \in \text{REG}$ and $h_i$ is an arbitrary homomorphism. We may assume that $L_i \subseteq \Sigma_i^*$ where all the $\Sigma_i$ are mutually disjoint alphabets. Let $R = \bigcup (R_i \cap \Sigma_i^*)$, $1 \leq i \leq n$ and let $h$ be the homomorphism defined by $h(\alpha) = h_i(\alpha)$ for each $\alpha$ in $\bigcup \Sigma_i$. It is straightforward to show that $\bigcup h_i(L_i' \cap R_i) = h((\bigcup L_i) \cap R)$, $1 \leq i \leq n$, which implies that $\hat{S}(K) \subseteq \Theta\Delta\{\bigcup L_i' \mid L_i' \in M(K); 1 \leq i \leq n\} = \Theta\Delta S(K)$.

$X = P$ (and similar for $X = F$): From 4.2 and 2.6 we obtain $\hat{P}(K) = \text{REG}/\Delta\Phi(K)/\text{FIN}$. Applying 2.3(2) twice yields: $\hat{P}(K) \subseteq \text{REG}/\Theta\Delta((\Phi(K) \cup \text{FIN})/\Phi(\text{FIN})) = \text{REG}/\Delta(\Phi(K)/\text{FIN}) \subseteq \Theta\Delta((\text{REG} \cup \text{FIN})/\Phi(\Phi(K)/\text{FIN})) = \Theta\Delta(\text{REG}/\text{FIN}) = \Theta\Delta P(K)$. □

Other Canonical Forms like $\hat{\chi}(K) = \hat{M}(K)/\text{REG}$ and $\hat{S}(K) = \hat{M}(K)/\text{ALPHA}$ (cf. [19, 20]) for $\sigma$-simple $K$, directly follow from 4.3.
In case \( X = C \) it is possible to establish a Canonical Form like \( \hat{C}(K) = \Pi C(K) = \Theta \Delta C(K) \) (cf. 4.3) provided \( K \) is \( \sigma \)-simple and \( L_0 \) is in \( K \) for each \( L \) in \( K \) and each symbol \( \alpha \) not occurring in any word of \( L \). In order to avoid further restrictions on \( K \) we will however follow another approach.

A substitution \( \tau : V \rightarrow K \) is called marked if \( \tau(\alpha) \subseteq \Delta^* \alpha \) for each \( \alpha \) in \( V \), and the alphabets \( \Delta \) and \( V \) are disjoint. Let \( K_1/K_2 \) denote the family obtained by applying marked \( K_1 \)-substitutions on languages from \( K_2 \), i.e., \( K_1/K_2 = \{ \tau(L) \mid L \in K_2 ; \tau \text{ is a marked } K_1 \text{-substitution} \}; \) cf. [19]. We now redefine the \( C \)-extension by \( C(K) = \text{REG}/(K/\text{STAR}) \); cf. 2.6. Notice that 4.1(2) remains valid. In the sequel we only consider this redefined extension, for we which we prove

**Theorem 4.4.** (Canonical Form Theorem for Full \( C \)-AFL’s.)

\( \hat{C} = \Pi C(K) = \Theta \Delta C(K) \).

**Proof.** Since \( \hat{C}(K) \) is closed under \( \Pi \) and \( C \), we have \( \Pi C(K) \subseteq \hat{C}(K) \). In order to establish the converse inclusion it suffices to show that \( \Pi C(K) \) is a full \( C \)-AFL containig \( K \).

Obviously, \( \Pi C(K) \) is a prequasoid and it is easy to prove that \( K \subseteq \Pi C(K) \). Thus it remains to show that \( \Pi C(K) \) is closed under \( C \). The proof is based on 2.2, 2.3(2) and the following inclusions (i) \( \Delta(K)/\text{STAR} \subseteq \Theta \Delta(K/\text{STAR}) \), (ii) \( (K/\text{STAR})/\text{STAR} \subseteq \Delta \Phi(K/\text{STAR}) \).

Let \( L \subseteq \Sigma^* \) be in \( K, R \in \text{REG} \) and let \( a, b \) and \( c \) be symbols not in \( \Sigma \). Define a homomorphism \( h \) and a finite substitution \( f \) by \( h(b) = a, h(\alpha) = \alpha \) for \( \alpha \in \Sigma \), and \( f(c) = \{a, ab\}, f(\alpha) = \{\alpha\} \) for \( \alpha \in \Sigma \). It is straightforward to prove that ((\( L \cap R)a)^* = h((Lb)^* \cap (Rb)^*) and ((La)^*b)^* = f((Lc)^* \cap (\Sigma \cup \{a, b\})^* b), which establish (i) and (ii), respectively.

Applying respectively 2.2, (i), 2.3(2), (ii) and 2.3(2) yields

\[
\begin{align*}
\Pi C(K) = \text{REG/}\Phi C(K)/\text{STAR} & \subseteq \text{REG/}\Theta \Delta(\Phi C(K)/\text{STAR}) \\
\Theta \Delta(\text{REG/}\Phi C(K)/\text{STAR}) & = \Pi(\text{REG/}(K/\text{STAR})/\text{STAR}) \\
\Pi(\text{REG/}\Phi(K/\text{STAR})) & \subseteq \Pi(\text{REG/}\Phi(K/\text{STAR})) = \Pi C(K),
\end{align*}
\]

i.e., \( \Pi C(K) \) is \( C \)-closed and hence \( \hat{C}(K) = \Pi C(K) \). Finally, we have

\( \Pi C(K) = \Theta \Delta(\text{FIN/REG/(K/\text{STAR})) = \Theta \Delta C(K) \). \( \square \)

From 4.2–4.4 we obtain

**Corollary 4.5.** Let \( X \) be a symbol from \( \{M,S,P,C,F\} \). Then

1. each full \( X \)-AFL contains the family \( X(\text{FIN}) \);
2. \( X(\text{FIN}) = \text{REG} \) is the smallest full \( X \)-AFL. \( \square \)

By \( X \)-AFL we will again denote the class of all full \( X \)-AFL’s over \( \Sigma_\omega \) (for \( X \in \{M,C,S,P,F\} \)).

**Theorem 4.6.**

1. \( M \text{-AFL} \supset S \text{-AFL} \supset P \text{-AFL} \supset F \text{-AFL} \supset R \text{-AFL}; \)
2. \( M \text{-AFL} \supset C \text{-AFL} \supset F \text{-AFL}; \)
3. \( C \text{-AFL} \) is incomparable with \( S \text{-AFL} \) and \( P \text{-AFL} \).
5 Substituting Families into Families

In this section we apply Canonical Forms of Sections 3 and 4 to the family obtained by substituting $K_1$-languages into languages from $K_2$ (5.1–5.3). Then we discuss a generalization of 4.2–4.4 and 5.2 (5.4–5.5) and finally, we consider an example of the application of Canonical Forms and related results in proving a certain family to be a full $X$-AFL (5.7).

We first consider substituting a prequasoid into a prequasoid.

Lemma 5.1. If $K_1$ is $\sigma$-simple and if $K_2$ is nontrivial, then

$$\Pi(K_1/K_2) \subseteq \Pi(K_1)/\Delta \Phi(K_2) = \Pi(K_1)/\Pi(K_2) = \Pi(K_1/\text{FIN}/K_2),$$

whereas equality holds whenever, either $K_1$ is closed under substitution into FIN (i.e., under union and concatenation), or $K_2$ is closed under finite substitution.

Proof. An application of 2.2 and 2.3(1) yields $\Pi(K_1/K_2) = \Theta \Delta(\Phi(K_1))/K_2 \subseteq \text{ONE}/\Delta \Phi(K_1)/\Delta \Phi(K_2) = \Pi(K_1)/\Delta \Phi(K_2) \subseteq \Pi(K_1)/\Pi(K_2)$. By 2.2 and 2.3(2) we have

$$\text{ONE}/\Delta \Phi(K_1)/\text{ONE}/\Delta \Phi(K_2) \subseteq \Theta \Delta(\Phi(K_1))/\Phi(\text{ONE})/\Delta \Phi(K_2) = \Theta \Delta(\Phi(K_1)/\text{FIN})/\Delta \Phi(K_2) \subseteq \Theta \Delta(\Phi(K_1))/\text{FIN}/\Phi(K_2) = \Pi(K_1/\text{FIN}/K_2).$$

Using the former inclusion we just established with FIN/K_2 instead of K_2, we obtain $\Pi(K_1/\text{FIN}/K_2) \subseteq \Pi(K_1)/\Delta \Phi(K_2)$. 

From 5.1 it directly follows that substituting a prequasoid into a prequasoid yields a prequasoid. A similar well-known conclusion for full trio, full semi-AFL, full AFL [9, 11] and also for full Kleene-AFL and full pseudo-AFL can be inferred immediately from

Theorem 5.2. Let $K_1$ and $K_2$ be $\sigma$-simple families. If $K_2$ is also closed under finite substitution, then

1. $\check{\mathcal{C}}(K_1/K_2) = \check{\mathcal{M}}(K_1)/\Delta(K_2/\text{STAR})$.
2. $\check{\mathcal{M}}(K_1/K_2) = \check{\mathcal{M}}(K_1)/\Delta(K_2)$.
3. $\check{\mathcal{S}}(K_1/K_2) = \check{\mathcal{M}}(K_1)/\Delta(K_2)/\text{ALPHA}$.
4. $\check{\mathcal{P}}(K_1/K_2) = \check{\mathcal{M}}(K_1)/\Delta(K_2)/\text{FIN}$.
5. $\check{\mathcal{F}}(K_1/K_2) = \check{\mathcal{M}}(K_1)/\Delta(K_2)/\text{REG}$. 

Proof. We will establish the existence of a full $C$-AFL $K_0$ which is not closed under union. Together with well-known results (cf. [9, 10]) this implies 4.6(1)–(3).

Let $K_1$ and $K_2$ be incomparable full AFL’s. (The existence of $K_1$ and $K_2$ is guaranteed by [13].) Let $L_1 \subseteq \Sigma_1^*$ be in $K_1 - K_2$ and let $L_2 \subseteq \Sigma_2^*$ be in $K_2 - K_1$ such that $\Sigma_1 \cap \Sigma_2 = \emptyset$. For $i = 1, 2$, define homomorphisms $h_i$ on $\Sigma_1 \cup \Sigma_2$ by $h_i(\alpha) = \alpha$ if $\alpha \in \Sigma_i$ and $h_i(\alpha) = \lambda$ if $\alpha \notin \Sigma_i$.

We define $K_0 = K_1 \cup K_2$, which is a full $C$-AFL [3]. Suppose $K_0$ is closed under union. Then according to [9, 10], $K_0$ is also closed under concatenation and, consequently, $L_1L_2 \in K_1$ or $L_1L_2 \in K_2$. But then we have $h_2(L_1L_2) = L_2 \in K_1$ or $h_1(L_1L_2) = L_1 \in K_2$, respectively, contradicting the choice of $L_1$ and $L_2$. 

□
Proof. (1) By 4.4 and 5.1 we have $\hat{\mathcal{C}}(K_1/K_2) = \Pi(\text{REG}/K_1/K_2/\text{STAR}) = \Pi(\text{REG}/K_1)/\Delta\Phi(K_2/\text{STAR})$, and hence by 4.3, $\hat{\mathcal{C}}(K_1/K_2) = \hat{\mathcal{M}}(K_1)/\Delta(K_2/\text{STAR})$.

(2) According to 4.2 and 5.1 we obtain $\hat{\mathcal{M}}(K_1/K_2) = \text{REG}/\Pi(K_1/K_2) = \text{REG}/\Pi(K_1)/\Delta\Phi(K_2) = \hat{\mathcal{M}}(K_1)/\Delta(K_2)$.

Together with 2.6 and 4.3 this implies (3)–(5).

In the remaining cases we even obtain

**Theorem 5.3.** Let $X$ be a symbol from \{R, A, H^*_1, H\}. Then

(1) $\hat{X}(K_1 \cup K_2) = X(\Pi(K_1) \cup \Pi(K_2)) = X(\Pi(K_1)/\Pi(K_2)) = X(\Pi(K_2)/\Pi(K_1))$.

(2) $\hat{X}(K_1/K_2) = \hat{X}(K_2/K_1) = \hat{X}(K_1 \cup K_2)$ provided both $K_1$ and $K_2$ are $\sigma$-simple.

Proof. (1) For reasons of symmetry it suffices to show the former two equalities. By 3.4(1) and [3] we have $\hat{X}(K_1 \cup K_2) = X(\Pi(K_1) \cup \Pi(K_2)) = X(\Pi(K_1)/\Pi(K_2))$. From 3.6 and 3.3(2) we obtain that each full $X$-AFL is a substitution-closed prequasoid (for $X \in \{R, A, H^*_1, H\}$), which implies $\Pi(K_1)/\Pi(K_2) \subseteq \hat{X}(K_1 \cup K_2)$. Since each prequasoid includes SYMBOL, we also have $K_1 \cup K_2 \subseteq \Pi(K_1)/\Pi(K_2) \subseteq \hat{X}(K_1 \cup K_2)$. Applying 3.4(1) and 5.1 (with $\Pi(K_1)$ instead of $K_i$, $i = 1, 2$) yields $\hat{X}(K_1 \cup K_2) = X(\Pi(K_1)/\Pi(K_2)) \subseteq X(\Pi(K_1)/\Pi(K_2)) \subseteq \hat{X}(K_1 \cup K_2)$.

(2) If both $K_1$ and $K_2$ are $\sigma$-simple, then $K_1 \cup K_2 \subseteq K_1 \cup K_2$. Hence $\hat{X}(K_1 \cup K_2) = \hat{X}(K_1/K_2)$.

We may also consider 4.2–4.4 and 5.2 as particular instances of more general results (cf. 5.4 and 5.5 below). We call a family $K$ a full $[K', K'']$-structure if $K$ is a prequasoid which is closed under $K'$-substitution and under substitution into $K''$, i.e., $K'/K/K'' \subseteq K$; cf. [19]. If $K'' = \text{STAR}$, we demand, as in Section 4, that $K$ is closed under marked substitution into $\text{STAR}$, i.e., $K'/K/\text{STAR}$. Let $[K', K''](K)$ denote the smallest full $[K', K'']$-structure containing $K$.

**Theorem 5.4.** Let $K'$ be a prequasoid and let $K''$ be either a prequasoid or equal to SYMBOL or ALPHA. Then

(1) $[K', \text{STAR}](K) = \Pi(K_{\infty}'/(K/\text{STAR})) = \Theta\Delta(K_{\infty}'/(K/\text{STAR}))$;

(2) $[K', K''](K) = K_{\infty}'/\Pi(K)/K_{\infty}'' = K_{\infty}'/\Delta\Phi(K)/K_{\infty}'' = \Pi(K_{\infty}'/K/K_{\infty}'') = \Theta\Delta(K_{\infty}'/K/K_{\infty}'')$, provided $K$ is $\sigma$-simple.

The proof of 5.4 consists of a straightforward modification of the arguments used in establishing 4.2–4.4 and it will therefore be omitted. Note that SYMBOL and ALPHA are substitution closed, i.e., $K_{\infty}'' \subseteq K''$.

From 5.1 and 5.4 we can infer in a way similar to 5.2

**Theorem 5.5.** Let $K'$ be a prequasoid and let $K''$ be either a prequasoid or equal to SYMBOL or ALPHA. If $K_1$ and $K_2$ are $\sigma$-simple families and, if $K_2$ is closed under finite substitution, then

(1) $[K', \text{STAR}](K_1/K_2) = \Pi(K_{\infty}'/K_1/K_2/\text{STAR}) = \Theta\Delta(K_{\infty}'/K_1/K_2/\text{STAR})$;

(2) $[K', K''](K_1/K_2) = K_{\infty}'/\Pi(K_1/K_2)/K_{\infty}'' = K_{\infty}'/\Pi(K_1)/\Delta(K_2)/K_{\infty}''$. 

\[\square\]
Taking $K'$ (and $K''$) equal to a quasoid instead of a prequasoid implies that $K'_\infty$ (and $K''_\infty$) is a full $R$-AFL (cf. 2.4) and that 5.4 and 5.5 yield full (pseudo-)AFL. In particular we have $\hat{\mathcal{P}}(K) = [\text{REG, FIN}](K)$ and $\hat{\mathcal{F}}(K) = [\text{REG, REG}](K)$, whereas full $M$-AFL, full $S$-AFL and full $C$-AFL correspond to the cases $K' = \text{REG}$, $K'' = \text{SYMBOL}$, $\text{ALPHA}$ and $\text{STAR}$, respectively (provided we apply a marked substitution in the last case).

When we take $K' = \text{FIN}$ instead of $\text{REG}$, we can obtain (1) prequasoids ($K'' = \text{SYMBOL}$), (2) prequasoids closed under union ($K'' = \text{ALPHA}$), (3) prequasoids closed under concatenation and union ($K'' = \text{FIN}$), (4) quasoids closed under Kleene $\star$ ($K'' = \text{STAR}$), and (5) quasoids closed under Kleene $\star$, union and/or concatenation ($K'' = \text{REG}$).

It is straightforward to show that results similar to 4.2–4.6 and 5.2 also hold for these structures.

Canonical Forms and theorems like 5.2–5.5 could serve as a useful tool in showing certain families to be a (particular kind of) full AFL or a full $[K', K'']$-structure. We conclude this paper with an application.

A language $L$ is called hyper-sentential [27] over a family $K$, if there exist (1) an alphabet $V$, (2) an initial language $L_0 \subseteq V^*$ in $K$, (3) a finite set $U$ of $K$-substitutions over $V$, such that $L = U^*(L_0)$. The hyper-sentential extension $\hat{\Sigma}(K)$ of $K$ consists of all languages hyper-sentential over $K$. Clearly, $L \cap \Sigma^*$ is in $H(K)$ when $L$ is in $\hat{\Sigma}(K)$. The hyper-algebraic extension $H(K)$ has been characterized in terms of $\hat{\Sigma}(K)$ by Van Leeuwen & Wood [27].

**Theorem 5.6.** Let $K$ be a family closed under isomorphism and under intersection with $\Sigma^*$ for each finite alphabet $\Sigma$. If $\text{ONE} \subseteq K$, then $H(K) = (\text{SYMBOL} \cup \{\lambda\})/\hat{\Sigma}(K)$. □

In the finite case even a stronger result holds: ETOL = SYMBOL/TOL and EOL = SYMBOL/OL (cf. [5, 6]; note that $\hat{\Sigma}(\text{FIN}) = \text{TOL}$). A SYMBOL-[or a SYMBOL $\cup \{\lambda\}$]-substitution is sometimes referred to as a [weak] coding.

We show how to obtain full AFL’s from $\hat{\Sigma}(K)$.

**Theorem 5.7.** Let $K$ be a full trio. If $K_0$ is a prequasoid, then $K/\hat{\Sigma}(K_0)$ is a $[\text{REG, } \hat{\mathcal{H}}(K_0)]$-structure and consequently, it is a full AFL.

**Proof.**

\[
\begin{align*}
[\text{REG}, \hat{\mathcal{H}}(K_0)](K/\hat{\Sigma}(K_0)) &= (K \text{ is a full trio}) \\
[\text{REG}, \hat{\mathcal{H}}(K_0)](K/(\text{SYMBOL} \cup \{\lambda\})/\hat{\Sigma}(K_0)) &= (5.6) \\
[\text{REG}, \hat{\mathcal{H}}(K_0)](K/H(K_0)) &= (5.5(2), 3.4) \\
\text{REG}/K/H(K_0)/H(K_0) &= (K \text{ is a full trio, } 3.4) \\
K/H(K_0) &= (5.6) \\
K/(\text{SYMBOL} \cup \{\lambda\})/\hat{\Sigma}(K_0) &= K/\hat{\Sigma}(K_0) (K \text{ is a full trio}),
\end{align*}
\]

i.e., $K/\hat{\Sigma}(K_0)$ is a $[\text{REG, } \hat{\mathcal{H}}(K_0)]$-structure. □

If $K$ is contained in $H(K_0)$, then clearly $K/\hat{\Sigma}(K_0) = H(K_0)$ and consequently it is a full $H$-AFL. Thus 5.7 is only interesting if $K$ is not included in $H(K_0)$. When $K_0 = \text{FIN}$, 5.7 yields: $K/\text{TOL}$ is a full AFL provided $K$ is a full trio, which was originally established
by Salomaa [23, 4]. Using \( EOL = \text{SYMBOL} / \text{OL} \) [5] one can prove in a way similar to 5.7 that \( K / \text{OL} \) is a full AFL whenever \( K \) is a full trio [23, 4].

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**References**

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**Notes**

The original typescript of this report consisted of 34 pages; the present LaTeX version reduced this number to 20.

One of the original references has been published (in slightly different form) as