A Market Model of Stochastic Smiles: a conditional density approach

Alex Zilber*

August 15, 2006

Abstract
We introduce a new approach that allows to construct no-arbitrage market models of implied volatility surfaces. The need for developing such models has long been recognized. The framework presented here makes it possible to generate models of a joint evolution of an arbitrary number of option price processes together with the underlying price process. The key idea of the approach is to take a deterministic smile model as a backbone around which a stochastic smile model can be constructed without violating no-arbitrage constraints.

1 Introduction
The problem of modelling an asset together with options written on this asset as tradable quantities in their own right has been faced by financial mathematics professionals for some time now. The need for such an approach is clear: this will simply reflect the reality better. Many option markets are mature enough to provide information which is not observable through the underlying markets. Numerous studies of the stochastic nature of implied volatility surfaces ([1],[8],[24],[11]) show that there are several sources of uncertainty that are not reflected in the underlying asset price. It is also clear that modelling this finer structure of the options market through classical approaches (such as local volatility, stochastic volatility, jump diffusion and their various mixtures and extensions) becomes extremely hard and yields models of ever-increasing complexity. Therefore, there is a clear need in developing a framework for modelling the stochasticity of implied volatilities directly.

A model where implied volatilities (or, equivalently, vanilla option prices) are modelled directly is called a market model. Potential applications of a market

---

*A recent work by Carr and Wu [7] is a fine example of a classical approach, where additional unobserved parameters are introduced in order to capture the stochasticity of the implied volatility skew in currency options markets.
model are in pricing heavily exotic and especially forward volatility dependent products, such as various compound options. Besides, a market model should allow for circumventing the calibration procedure (since the observed implied volatility surface is included in the model's initial state) and more natural hedging with vanilla instruments. Finally, the ability of such a model to generate no-arbitrage paths for future spot and option prices should prove useful in risk management.

There have been many attempts at developing a satisfactory market model. To the best of our knowledge, one of the first works on the subject is by Schönbucher[23]. Other papers include those by Cont et al[9], Ledoit et al[17], Brace et al[5], Alexander and Nogueira[2], Häfner and Schmid[14] Fengler et al, [12], Le[16] and others. We deem a careful review of all these papers outside the scope of this article.

2 Outline and a sketch explanation

In this very short section we give an outline of the paper and a sketch of the main idea. If the explanation appears too vague, one can skip directly to the next section. On the other hand, should the formalism introduced later become unclear and leading nowhere, it might help to check with this section again in order to see the bigger picture.

As explained in the introduction, the goal is to build a market model of implied volatilities (or option prices). Hence, unlike in the classical models, the state space can not be limited to an asset price and, possibly, some unobserved parameters (e.g. stochastic volatility). On the contrary, it has to be extended with market quotes of implied volatilities or with option prices for different strikes and maturities. This new extended state space is introduced in section 3. In the same section we formulate the main theorem, which states that in this new space a stochastic process can be defined that would model a joint no arbitrage evolution of asset and option prices.

However, one can not introduce a stochastic process unless a probability space has been defined. Thus, constructing a probability space and a stochastic process living on it, is the crux of the paper and is presented in section 4. The probability space is built by virtue of a Kolmogorov theorem. We introduce a Kolmogorov-compatible family of measures that gives a rise to a probability space and a stochastic process. In turn, the measures are constructed through transition density functions defined in the state space. The trick here is to build transition functions in a one-dimensional space (where an asset price lives) first and only then build full-space transition functions as some sort of superstructure on top of them. This way it is easy to ensure that the resulting price processes satisfy the martingale property. From the martingale property it is just one step to the no arbitrage condition, applying a fundamental asset pricing theorem.

Next, section 5 presents a meta-algorithm for simulation, based on the theoretical framework developed in the paper. This section is meant to give an idea of how the approach can be used in practice.
Finally, the last section 6 concludes the paper and discusses possible directions for further research.

3 Basic notations and the main statement.

3.1 The state space, admissibility conditions, the market.

Definition 3.1 The state space is $D = \mathbb{R}^d$.
The state variable is $x_t = \{s_t, \theta_t\}$,
t $\in I$, where $I \subset \mathbb{R}_+$ is the time domain,
s $\in \mathbb{R}$ - prevailing asset price at $t$,
$\theta_t \in \mathbb{R}^{d-1}$ - finite dimensional vector parametrizing the smile surface at time $t$.
Altogether, we have $x_t \in D, t \in I$.

One can think of $\theta_t$ as a collection of option quotes available in the market.

In the FX context one can set

$\theta_t = \left\{ \sigma_{t, \text{atm}}^{\text{ld}}, \sigma_{t, 25\Delta \text{rr}}^{\text{ld}}, \sigma_{t, 25\Delta \text{str}}^{\text{ld}}, \ldots; \sigma_{T_n, \text{atm}}^{\text{ld}}, \sigma_{T_n, 25\Delta \text{rr}}^{\text{ld}}, \sigma_{T_n, 25\Delta \text{str}}^{\text{ld}} \right\}$, 

where a superscript denotes the maturity of the quote in question\(^2\).

We further assume that a (second-order) continuous implied volatility surface $I_t(K,T)$ can be obtained given $x_t = \{s_t, \theta_t\}$. In other words, there is a functional

$A : \mathbb{R}^d \to C^2_{\mathbb{R}^2}$

$A(x_t) = I_t(\cdot, \cdot)$

Not every $C^2$-function can defines an non-arbitrageable implied volatility surface. Sufficient conditions are formulated in this section.

First, let us define

$C_{\text{BS}} : (t, s, K, T, \sigma) \to \mathbb{R}$

(4)
to be a standard Black-Scholes call option price\(^3\) function (for the sake of brevity we suppress any dependence on interest rates and dividends in our notation).

Then, further define

$C(x_t, K, T) = C_{\text{BS}}(t, s_t, K, T, A(x_t)(K, T))$

(5)

\(^2\)The exact meaning of an at-the-money volatility quote, a 25-delta risk reversal quote and a 25-delta strangle quote is left outside the scope of this paper. The reader is referred to [19] for definitions and to [4] for qualitative discussion.

\(^3\)Here and everywhere throughout the paper, ‘call price’ is short for undiscounted option price’, unless explicitly stated otherwise.
Definition 3.2 The state $x_t$ (and volatility surface $I(K,T) = A(x_t)(K,T)$) is called admissible iff

\[ I(K,T) > 0 \]
\[ \frac{\partial C(x_t, K, T)}{\partial K} < 0 \]
\[ \frac{\partial^2 C(x_t, K, T)}{\partial K^2} > 0 \]
\[ \frac{\partial C(x_t, K, T)}{\partial T} > 0 \]
\[ \frac{\partial P(x_t, K, T)}{\partial K} > 0 \]
\[ C(x_t, K, T)|_{K=0} = s_t \]
\[ \lim_{K \to \infty} C(x_t, K, T) = 0 \]

As shown in [15], the above admissibility conditions are sufficient to exclude static arbitrages for $I(K,T)$.

3.2 The main statement

As usual, we will assume a perfect frictionless market, where an asset and its derivatives are liquidly traded. Current state of the market is a collection of the underlying and options quotes, which can be represented by an admissible (as in definition 3.2) point in our state space:

Definition 3.3 An admissible point $x_0$ in the state space $\mathcal{D}$ is called an initial market state, if $x_0 = (s_{\text{market}}, \theta_{\text{market}})$, where $s_{\text{market}} \in \mathbb{R}$ - currently prevailing asset price in the market, $\theta_{\text{market}} \in \mathbb{R}^{d-1}$ - a collection of currently prevailing implied volatility (and/or option price) quotes.

In the FX context, $\theta_{\text{market}}$ could be the collection of standardized market quotes:

\[ \{ \sigma_{\text{market, atm}}^{T_1}, \sigma_{\text{market, 25\Delta, rr}}^{T_1}, \sigma_{\text{market, 25\Delta, str}}^{T_1}, \cdots, \sigma_{\text{market, 25\Delta, rr}}^{T_n}, \sigma_{\text{market, 25\Delta, str}}^{T_n} \} \]

We would like to be able to model a no-arbitrage evolution of an asset price and an associated implied volatility surface through a stochastic process, taking values in $\mathcal{D}$. In this paper we show how to do it when the process in question is a discrete-time one. So let us assume that the time domain is finite or infinitely discrete: $I \subset \mathbb{N}_0$, and, without the loss of generality, that $0 \in I$. The following is the formulation of our main result.

Theorem 3.4 Let $x_{m}$ be an initial market state, as defined in 3.3. Let $I$ be a finite or infinite subset of $\mathbb{N}_0$. Then there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in I}, P)$ and a $\mathcal{D}$-valued stochastic process $(X_t)_{t \in I}$, $P$, such that it
makes a no-arbitrage market model of asset and implied volatilities (option prices) and, moreover, \( X_0 = x_m \) (therefore, the model is calibrated to the currently observed market information).

We present a constructive proof of this theorem (see section 4) and, consequently, a meta-simulation algorithm with it.

4 Proof of the main statement
This section contains a constructive proof of Theorem 3.

4.1 Transition functions

**Definition 4.1** A family of functions \((\rho_{t_1,t_2}(s_1,s_2))_{t_1,t_2 \in I}\) acting on \(\mathbb{R} \times \mathbb{R}\) is called an s-transition family (and its members are called s-transition functions) if functions \(\rho_{t_1,t_2}(s_1,s_2)\) satisfy the following conditions:

\[
\rho_{t_1,t_2}(s_1,s_2) \geq 0, \tag{7}
\]

\[
\int \rho_{t_1,t_2}(s_1,s_2)ds_2 = 1 \tag{8}
\]

\[
\rho_{t_0,t_2}(s_0,s_2) = \int \rho_{t_1,t_2}(s_1,s_2)\rho_{t_0,t_1}(s_0,s_1)ds_1, \tag{9}
\]

where each of the equations has to hold for all \(t_0, t_1, t_2 \in I\) and \(s_0, s_1, s_2 \in \mathbb{R}\).

**Definition 4.2** We say that an s-family \((\rho_{t_1,t_2}(s_1,s_2))_{t_1,t_2 \in I}\) is calibrated to a state \(x_m \in D\) at time \(t\) if

\[
\rho_{t,T}(s_m,s) = \left. \frac{\partial^2 C(x_m,K,T)}{\partial K^2} \right|_{K=s}, \text{ for all } K, T \in \mathbb{R}_+, \tag{10}
\]

where \(C(x_m,K,T) = C_{BS}(t,s_m,K,T,A(x_m)(K,T))\).

For the sake of brevity, we will sometimes say ‘s-family \((\rho_{t_1,t_2}(s_1,s_2))\) is \(x_m\)-calibrated’.

Note that the problem of finding a \(x_0\)-calibrated family of s-transition functions for a given admissible \(x \in D\) at a given time \(t_0\) does not have a unique solution. Indeed, as discussed in a number of papers (see, for example [3], [4], [15]) condition (10) only fixes the terminal members of an s-family: all functions \(\rho_{t_0,t_2}(s_0,s_2)\) for a given pair of \(t_0, s_0\) and all \(t_2 \in I, s_2 \in \mathbb{R}_+\). However, for any \(t_1 > t_0\), the function \(\rho_{t_1,t_2}(s_1,s_2)\) has to be found from the integral equation (9), which has infinitely many solutions.

Let us assume that there is an algorithm at our disposal that solves this problem uniquely for each admissible state \(x \in D\). A number of such algorithms can be found. One rather straightforward procedure is described in [20].

So let us fix a map
If the functions \( R \) called an x-transition family (and its members are called x-transition functions) a family of functions

\[
B(x, t) = (\rho_{t_1, t_2}(s_1, s_2))_{t_1, t_2 \in I},
\]

where \((\rho_{t_1, t_2}(s_1, s_2))\) is calibrated to \( x \) at time \( t \).

Sometimes, we will also use \((\rho_{t_1, t_2}^{x,t}(s_1, s_2))\) to denote that this s-family of transition functions is obtained from \( x, t \) by applying the algorithm \( B(\cdot, \cdot) \).

We will also need an analogue of an s-transition function in our \( d \)-dimensional state space:

**Definition 4.3** A family of functions \((R_{t_1, t_2}(x_1, x_2))_{t_1, t_2 \in I}\) acting on \( D \times D \) is called an x-transition family (and its members are called x-transition functions) iff the functions \( R_{t_1, t_2}(x_1, x_2) \) satisfy the following conditions:

\[
R_{t_1, t_2}(x_1, x_2) \geq 0, \tag{12}
\]

\[
\int R_{t_1, t_2}(x_1, x_2) dx_2 = 1 \tag{13}
\]

\[
R_{t_0, t_2}(x_0, x_2) = \int R_{t_1, t_2}(x_1, x_2) R_{t_0, t_1}(x_0, x_1) dx_1, \tag{14}
\]

where each of the equations has to hold for all \( t_0, t_1, t_2 \in I \) and \( x_0, x_1, x_2 \in D \).

**Definition 4.4** An x-transition family \((R_{t_1, t_2}(x_1, x_2))_{t_1, t_2 \in I}\) is admissible iff

\[
R_{t_1, t_2}(x_1, x_2) = 0 \text{ whenever } x_1 \text{ is admissible and } x_2 \text{ is not admissible}, \tag{15}
\]

for all \( t_1, t_2 \in I \).

Since \( x_t = (s_t, \theta_t) \), sometimes it will be more convenient to use a slightly different notation

\[
R'_{t_1, t_2}(s_1, \theta_1, s_2, \theta_2) = R_{t_1, t_2}((s_1, \theta_1), (s_2, \theta_2)) = R_{t_1, t_2}(x_1, x_2) \tag{16}
\]

We can also factor \( R'_{t_1, t_2}(s_1, \theta_1, s_2, \theta_2) \) into a product of two functions, as follows:

\[
R'_{t_1, t_2}(s_1, \theta_1, s_2, \theta_2) = r^s_{t_1, t_2}(s_1, \theta_1, s_2) \cdot r^\theta_{t_1, t_2}(s_1, \theta_1, s_2, \theta_2), \tag{17}
\]

Where the functions on the right hand side are defined as follows

\[
r^s_{t_1, t_2}(s_1, \theta_1, s_2) := \int R'_{t_1, t_2}(s_1, \theta_1, s_2, \theta_2) d\theta_2, \tag{18}
\]

\[
r^\theta_{t_1, t_2}(s_1, \theta_1, s_2, \theta_2) := \frac{R'_{t_1, t_2}(s_1, \theta_1, s_2, \theta_2)}{r^s_{t_1, t_2}(s_1, \theta_1, s_2)}. \tag{19}
\]
Definition 4.5 We say that an $x$-family $(R_{t_1,t_2}(x_1, x_2))_{t_1,t_2 \in I}$ is a superstructure of an $s$-family $B(x_0, t_0) = (\rho_{t_1,t_2}^{s_1,s_2}(s_1, s_2))_{t_1,t_2 \in I}$ (or, equivalently, the latter is the base of the former) iff the following two conditions are met:

$$\int C(x_1, K, T)r_{t_0,t_1}(s_0, \theta_0, s_1, \theta_1)d\theta_1 = \int (S_T - K)^+ \rho_{t_1,T}^{x_0,t_0}(s_1, s_T)dS_T$$  \hfill (20)

for all $K, T \in \mathbb{R}_+$. 

$$r_{t_0,T}^s(s_0, \theta_0, s_T) = \rho_{t_0,T}^{x_0,t_0}(s_0, s_T) \text{ for all } T, s_T \in \mathbb{R}_+.$$  \hfill (21)

This definition seems a bit odd, but its meaning will become quite clear later.

Definition 4.6 We say that an $x$-family $(R_{t_1,t_2}(x_1, x_2))_{t_1,t_2 \in I}$ is calibrated to a state $x_m \in \mathcal{D}$ at time $t$ iff it is a superstructure of an $s$-family $(\rho_{t_1,t_2}(s_1, s_2))$, which is calibrated to $x_m$ at $t$.

### 4.2 Distributions and probability space

We will now construct a family of measures that will form a base for our probability space. Intuitively, the procedure is quite simple. In the case of a discrete time domain $I$, think of $x$-transition functions $R_{t_k,t_{k+1}}(x_k, x_{k+1})$ as of transition densities of a discrete-time process taking values in $\mathcal{D}$. Given these density functions it is easy to define a joint distribution of the process values at any finite set of time points. From a family of distribution there is just one step to constructing a probability space, by virtue of the Kolmogorov extension theorem.

However, for the sake of completeness and clarity, we would like to provide a rather detailed description of this construction, which follows immediately.

We are still working under the assumption that $I$ is finite or infinitely discrete. It is also assumed that $I \subset \mathbb{N}_0$ and $0 \in I$.

Consider a family of measurable spaces $\{D_i, B_i\}$, where $D_i = \mathbb{R}^d$, $B_i$ - Borel sets on $\mathbb{R}^d$ and $i \in I$.

We will construct a family of measures $\{m_{i_1,i_2,\ldots,i_n}; n = 1, 2, \ldots; i_k \in I\}$, such that $m_{i_1,i_2,\ldots,i_n}$ is a measure on $\prod_{k=1}^n \{X_{i_k}, B_{i_k}\}$.

We define $m_{i_1,i_2,\ldots,i_n}$ as follows. First, let us introduce some notation. For the sake of brevity, we will denote a finite sequence of indices $i_1, i_2, \ldots, i_n$ as $\bar{i}$. Also, let us write $M_{\bar{i}}$ for a maximum index in $\bar{i}$ which does not correspond to a trivial (equal to $\mathbb{R}^d$ itself) Borel set

$$M_{\bar{i}} = \max \{j : j \in \bar{i} \text{ and } B_j \neq X_j\}.$$  \hfill (22)
Definition 4.7 A sequence of Borel sets $\mathcal{C} = \{C_1, C_2, \ldots, C_M\}$ is called the canonical sequence for $i$ if and only if it satisfies the following two conditions:

\[
C_j = B_{i_k} \text{ in case } j \in \bar{i} \text{ and } j = i_k 
\]  
\[
C_j = X_j \text{ if } j \notin \bar{i} 
\]  

In other words, we sort $\{B_{i_1}, B_{i_2}, \ldots, B_{i_n}\}$ into a sequence with ascending indeces, going up to the last non-trivial index $i < M$, that is, for every index $i < M$ missing in the sequence $\{i_1, i_2, \ldots, i_n\}$. The resulting sequence is Canonical for $\bar{i}$, which is denoted as $\mathcal{C}(\bar{i})$.

For any index sequence $\bar{i}$ we define $m_{i_1,i_2,\ldots,i_n}(\prod_{k=1}^{n} B_k)$ through its canonical sequence of Borel sets $\mathcal{C}(\bar{i})$.

Definition 4.8 Set

\[
m_{i_1} \left( \prod_{k=1}^{n} B_{i_k} \right) := F(\mathcal{C}(\bar{i})), \text{ where} 
\]

\[
F(\mathcal{C}) = \int_{C_{i_1}} \ldots \int_{C_{i_n}} R_{t_{k-1},t_k}^{x_{k-1},x_k}(x_0,x_1) \ldots R_{t_{n-1},t_n}^{x_{n-1},x_n}(x_{n-1},x_n) dx_1 \ldots dx_n 
\]  
\[
(26)
\]

and each $R_{t_{k-1},t_k}^{x_{k-1},x_k}(x_{k-1},x_k)$ is a $x$-transition from a $x$-family $x_{k-1}$-calibrated family.

The resulting $m_{i_1,i_2,\ldots,i_n}$ is a family of measures.

If for each $k$ the family $\left(R_{t_{k-1},t_k}^{x_{k-1},x_k}(x_1,x_2)\right)$ is admissible and a superstructure of an $s$-family

\[
B(x_{k-1},t_{k-1}) = \left(\rho_{t_{k-1},t_k}(s_1,s_2)\right)_{t_1,t_2 \in I}, 
\]

then $m_{i_1,i_2,\ldots,i_n}$ is called a $x$-generated family of measures.

Considering that $R_{t_{k-1},t_k}(x_{k-1},x_k)$ is an $x$-transition function and as such satisfies all technical conditions necessary for a density functions (non-negativity, integrability to one), we can show that $F(\mathcal{C})$ is indeed a measure.

To see this, consider the case when $C_j = (-\infty, c_{j,1}) \times (-\infty, c_{j,2}) \times \cdots \times (\infty, c_{j,k+1})$. Acting on such $C$’s, $F(\cdot)$ clearly is a distribution function (follows from the technical conditions imposed on $R_{t_{k-1},t_k}(x_{k-1},x_k)$). These distribution functions give rise to a measure $F(\mathcal{C})$ as defined above.

Proposition 4.9 The family of measures $\{m_{i_1,i_2,\ldots,i_n}; n = 1, 2, \ldots; i_k \in I\}$ defined above satisfies the following two conditions:

\[
m_{i_1,\ldots,i_{n+r}} \left( \prod_{k=1}^{n+r} B_{i_k} \right) = m_{i_1,\ldots,i_n} \left( \prod_{k=1}^{n} B_{i_k} \right), 
\]

(27)
if \( B_{ij} = X_{ij} \) for \( j = n + 1, \ldots, n + r \).

\[
m_{i_1, \ldots, i_n} \left( \prod_{k=1}^n B_{ik} \right) = m_{i_{s_1}, \ldots, i_{s_n}} \left( \prod_{k=1}^n B_{i_{s_k}} \right),
\]

where \( \{s_1, s_2, \ldots, s_n\} \) is some permutation of \( \{1, 2, \ldots, n\} \).

**Proof** In view of (23) and (24) canonical forms are immune to permutations and to adding trivial (equal to \( R^d \)) Borel sets. Hence, both in (27) and (28) the measures on the left and on the right represent multidimensional integrals over the same product of Borel sets, defined by the canonical sequence which is identical for the arguments on the left and right hand sides. ■

In fact, (27) and (28) can be recognised as Kolmogorov’s compatibility conditions for a family of measures. Hence, we have \( D_i - \) complete, separable metric spaces (in our case, \( D_i = \mathbb{R}^d \)). There are \( B_i - \) \( \sigma \)-algebras of Borel sets on \( D_i \). There is a family of measures \( \{m_{i_1, i_2, \ldots, i_n}; n = 1, 2, \ldots; i_k \in I\} \) which is defined on finite products of \( B_k \in B_k \) and which satisfies Kolmogorov’s compatibility conditions. Hence, by Kolmogorov extension theorem (see [13]), there is a probability space \( \{\Omega, \mathcal{F}, P\} \) and a stochastic process \( (X_i)_{i \in I} \) such that

\[
m_{i_1, i_2, \ldots, i_n} \left( \prod_{k=1}^n B_{i_k} \right) = P \left( \bigcap_{k=1}^n \{X_{i_k} \in B_{i_k}\} \right)
\]

Let us use \( (\mathcal{F}_i)_{i \in I} \) to denote a filtration generated by the process \( (X_i)_{i \in I} \). Obviously, \( \mathcal{F}_i \subset \mathcal{F} \) for all \( i \in I \) and \( (X_i) \) is adapted to the filtration \( (\mathcal{F}_i) \).

**Definition 4.10** A combination of a filtered probability space \( \{\Omega, \mathcal{F}, (\mathcal{F}_i)_{i \in I}, P\} \) and a stochastic process \( (X_i)_{i \in I} \), constructed according to the definitions above (e.g. based on a \( x \)-generated family of measures), is called a market model of stochastic smiles.

### 4.3 Martingale property

In this section we will show that in the above-introduced probabilistic model all price processes satisfy a martingale property. That is to say, for any \( X_i = \{S_t, \Theta_t\} \) option prices \( C(X_i, K, T) = C_{BS}(t_i, S_{t_i}, K, T, A(X_i)(K, T)) \) and the asset price \( S_i \) are martingales w.r.t. \( (\mathcal{F}_i)_{i \in I} \).

**Proposition 4.11** Let \( \{\Omega, \mathcal{F}, (\mathcal{F}_i)_{i \in I}, P\} \) and \( (X_i)_{i \in I} \) be a market model of stochastic smiles (e.g. based on a \( x \)-generated family of measures). Then both asset and option prices are martingales with respect to \( (\mathcal{F}_i) \):

\[
E \left( S_t | \mathcal{F}_s \right) = d_{s,t}^{-1} S_s \quad \text{for any } t, s \in I, \text{ such that } s < t.
\]

\[
E \left( C(X_i, K, T) | \mathcal{F}_s \right) = C(X_s, K, T) \quad \text{for any } t, s \in I, \text{ such that } s < t,
\]

and for any \( K, T \in \mathbb{R}_+ \).

Here \( d_{s,t}^{-1} \) denotes the discount factor from time \( s \) to time \( t \) (remember that option prices are undiscounted, hence no discount factor in the second equation).
Proof Note that in since we have a discrete time domain \( I = t_0, t_1, \ldots \) in order to prove (30) and (31) it suffices to show

\[
E(S_{k+1} | \mathcal{F}_k) = d\int_{t_k}^{t_{k+1}} S_k \text{ for any } k \in \mathbb{N}_0, \tag{32}
\]

\[
E(C(X_{k+1}, K, T) | \mathcal{F}_k) = C(X_k, K, T) \text{ for any } k \in \mathbb{N}_0, \tag{33}
\]

and for any \( K, T \in \mathbb{R}_+ \).

Remember, that our measure was defined through \( x \)-transition functions \( R_{t_k, t_{k+1}}(x_k, x_{k+1}) \). By construction, these functions are also transition density functions of the process \( (X_i) \) under measure \( P \). Using this fact we can rewrite the expectation in (33) as follows:

\[
E(C(X_{k+1}, K, T) | X_k)) = \int C(x_{k+1}, K, T) R_{t_k, t_{k+1}}(X_k, x_{k+1}) dx_{k+1} \tag{34}
\]

Now we can finally use all the properties of an \( x \)-transition function. Remember also, that since our model is based on a properly generated measure, the state density family \( R_{t_k, t_{k+1}}(x_k, x_{k+1}) \), on a \( k \)-th step, is a superstructure of a \( x_k \)-calibrated \( s \)-family

\[
B(x_k, t_k) = \left( \rho_{t_k, t_{k+1}}^{x_k,t_k}(s_k, s_{k+1}) \right)
\]

Using the properties of superstructures, the expectation can be further rewritten as follows:

\[
E \left( C(X_{k+1}, K, T) | X_k \right) = \int C(x_{k+1}, K, T) R_{t_k, t_{k+1}}(S_k, \Theta_k, s_{k+1}, \theta_{k+1}) d\theta_{k+1} ds_{k+1}
\]

\[
= \int C(x_{k+1}, K, T) r^{s}_{t_k, t_{k+1}} (S_k, \Theta_k, s_{k+1}) \cdot r^{\theta}_{t_k, t_{k+1}} (S_k, \Theta_k, s_{k+1}, \theta_{k+1}) d\theta_{k+1} ds_{k+1}
\]

\[
= \int (s_T - K)^+ \cdot \rho_{t_k, t_{k+1}, T}^{X_k, t_k, t_{k+1}}(s_{k+1}, s_T) ds_T \cdot r^{s}_{t_k, t_{k+1}} (S_k, \Theta_k, s_{k+1}) ds_{k+1}
\]

\[
= \int (s_T - K)^+ \rho_{t_k, T}^{X_k, t_k}(S_k, s_T) \cdot \rho_{t_k, t_{k+1}}^{X_k, t_k}(s_{k+1}, s_T) ds_T = C(X_k, K, T)
\]

Explanation: second to third line: factorization (17) of \( x \)-transition functions, third to fourth line: (4.5), condition one, fourth to fifth line: (4.5), condition two, fifth to sixth line: density convolution, holds in view of (4.1), last step, line six: follows from the fact that \( \rho_{t_k, t_{k+1}}^{x_k,t_k}(s_1, s_2) \) is \( x_k \)-calibrated at
time $t_k$. 

This completes the proof of (33). It remains to prove (32), which is done in a similar way. First, a completely analogous derivation as above yields the following

$$E(S_{k+1} | F_k) = \int s_{k+1} \rho_{t_k, t_{k+1}} (S_k, s_{k+1}) ds_{k+1}$$  \hspace{1cm} (35)

It only remains to be shown that the integral on the right equals the forward spot price. This property holds, since we are integrating the future asset price against a function which is essentially a risk-neutral asset price density observed in the market. To see this consider the following. First, $X_k$ is admissible, which follows from the admissibility of all transition families and from the fact that we started from an admissible state $x_0$. By construction, $\rho_{t_1, t_2} (s_1, s_2)$ is $X_k$-calibrated and hence, by (4.2) and (3.2), we can substitute the $s$-transition function on the right hand side of (35) by the second derivatives of option prices at $X_k$. Next, we integrate by parts twice and arrive at the desired result:

$$\int s_{k+1} \rho_{t_k, t_{k+1}} (S_k, s_{k+1}) ds_{k+1} = Df_{t_k, t_{k+1}}^{-1} S_k,$$  \hspace{1cm} (36)

From this (32) immediately follows and the whole proposition proof is complete.

4.4 Proof of the main statement

Armed with all the preliminary results obtained in the previous sections, we can now prove the main result, Theorem (3.4). The point of this theorem is that given a statically not arbitrageable market, that is an asset price and implied volatility (option prices) quotes, we can present a no-arbitrage stochastic framework, where the asset and option prices are evolved simultaneously.

Obviously, the candidate for such a model is the market model of stochastic smiles, introduced above. Let us formulate the no arbitrage condition in this context.

Let $x_0$ at time 0 be an initial market state. Let $\{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in I}, P\}$ and $(X_t)_{t \in I}$ be a market model of stochastic smiles (based on an $x$-generated family of measures $m_{i_1, i_2 \ldots i_l}$). We set $X_0 = x_0$ and use a $x_0$-calibrated $x$-transition family $R_{0, t_1} (x_0, x_1)$, when constructing $m_{i_1, i_2 \ldots i_l}$. Let $\{K_i, T_i\}, i = 1, \ldots, n-1$ be an arbitrary selection of strike and maturity pairs. Define the corresponding $n$-dimensional stochastic vector process $Y$ as

$$(Y_t)_{t \in I} = \left( [S_t, C(X_t, K_1, T_1), \ldots, C(X_t, K_n, T_n)]^T \right)_{t \in I}$$  \hspace{1cm} (37)

The no-arbitrage condition can be formulated as follows (here we follow [22]):
Definition 4.12 For any \( n \in \mathbb{N} \) and any selection of \( \{K_i, T_i\}, i = 1, \ldots, n - 1 \), consider the corresponding vector of tradables \( (Y_t)_{t \in I} \), defined as in (37). Let \( K_0 \) be the vector space of easy stochastic integrals:

\[
K_0 = \text{span} \{ h(w), Y_t(w) - Y_s(w) \},
\]

where \( w \in \Omega \), \((\cdot, \cdot)\) denotes the inner product in \( \mathbb{R}^n \), \( s, t \) runs through the pairs in \( I \) with \( s < t \) and \( h \) is an \( \mathbb{R}^n \)-valued \( \mathcal{F}_s \)-measurable function. Note that \( K_0 \) is a subspace of \( L^0(\Omega, \mathcal{F}, P) \), which is a space of all \( \mathcal{F} \)-measurable real-valued functions.

We say that \( (Y_t)_{t \in I} \) satisfies the no-arbitrage condition (NA) if

\[
K_0 \cap L^0_+ = 0.
\]

Here \( L^0_+ \) stands for all \( P \)-a.s. positive functions from \( L^0(\Omega, \mathcal{F}, P) \).

The economic interpretation of (NA) is as follows: \( K_0 \) represents a subspace of trading strategies based on the simple trading operations (random variables) of the form \( (h(w), Y_t(w) - Y_s(w)) \). Such a random variable represents the net gain of buying \( h(w) \) units of asset and options at time \( s \) and selling them again at time \( t \). At the moment of buying an agent can only use information modelled by \( \mathcal{F}_s \). Hence, the (NA) says that no zero-initial capital strategy based on trading at times from \( I \) yields a positive expected profit with zero risk.

Also note that Proposition 4.11 gives us the existence of an equivalent martingale measure condition (EMM). There is a whole collection of results that link the no-arbitrage conditions with existence of an equivalent martingale measure condition (EMM). These results, also known as fundamental theorems of asset pricing, are summarized in Delbaen and Schachermayer[10] and we refer the reader there for more details. For our purposes it suffices to say that under assumptions and notations introduced above, the implication (EMM) \( \Rightarrow \) (NA) is elementary and the proof can be found, for example [22]\(^4\).

We can therefore conclude that the market model of stochastic smiles yields satisfies (NA) and hence we have obtained a no arbitrage model of an asset and its associated options prices, as defined in (4.12). This completes the proof of Theorem (3.4).

5 Simulation

In this section we provide a meta-algorithm for a simulation based on the theoretical framework introduce above. For the sake of demonstration we set the example in the FX context.

Imagine that we are pricing an option, whose payoff is a function \( P(\cdot, \cdot) \) of a spot price and an implied volatility surface at some future point of time \( t_e \):

\(^4\)In our case (EMM) also implies stronger forms of the no-arbitrage conditions, known as ‘no free lunch’ and ‘no free lunch with bounded risk’, but for the sake of brevity we leave these notions outside of the scope of this article and refer the reader to [22] yet once again.
Assume that there is market data information encapsulated in the variable $x_0 = \{s_0, \theta_0\}$, where

$$\theta_0 = \left\{ \sigma_{t_0,\text{atm}}, \sigma_{t_0,25\Delta,\text{rr}}, \sigma_{t_0,25\Delta,\text{str}}; \ldots; \sigma_{t_n,\text{atm}}, \sigma_{t_n,25\Delta,\text{rr}}, \sigma_{t_n,25\Delta,\text{str}} \right\}.$$  

We select some algorithm $A(\cdot)$ for interpolating the implied volatility surface. There are plenty of numerically cheap and accurate algorithms for that so we will not discuss it here.

Note, however, that the choice of the algorithm $B(\cdot, \cdot)$ includes fixing a method for getting terminal densities from option prices, which is pretty standard, and a method for prescribing conditional transition densities in spot dimension. This is less standard and one needs to make a thoughtful choice here. We leave a detailed discussion outside the scope of this paper, but refer the reader to [20], where one such method is described.

Let us set a simulation step $\delta t$ and denote $X_i = (S_i, \Theta_i) = (S_{t_i}, \Theta_{t_i})$ where $t_i = t_0 + i\delta t$. Also, we fix the choice of tenors $T_1, \ldots, T_n$ to evolve and the standardized market quotes $\sigma_{t,\text{atm}}, \sigma_{t,25\Delta,\text{rr}}, \sigma_{t,25\Delta,\text{str}}$ per tenor.

Here is the meta-algorithm in steps.

**Start of the algorithm.**

**Step 1:**

Extract s-transition densities from the currently observed market data:

$$\left( \rho_{t_1,t_2}^{x_0}(s_1, s_2) \right)_{t_1, t_2 \in I} = B(x_0, 0)$$  

(41)

**Step 2:**

Sample $s_1 = S_1(\omega)$ from a distribution given by $\left( \rho_{t_0,t_1}^{x_0}(s_0, s_1) \right)$.

**Step 3:**

Convert the set of transition densities in s-dimension into an implied volatility surface and a collection of corresponding implied quotes:

$$\rho_{t_i, T_i}(s_1, s_i) \rightarrow \left\{ \sigma_{t_i,\text{atm}}, \sigma_{t_i,25\Delta,\text{rr}}, \sigma_{t_i,25\Delta,\text{str}} \right\}, i \in \{1, \ldots, N\}$$  

(42)

Denote $\theta_{s_1} = \left\{ \sigma_{t_1,\text{atm}}, \sigma_{t_1,25\Delta,\text{rr}}, \sigma_{t_1,25\Delta,\text{str}}, \ldots, \sigma_{t_n,\text{atm}}, \sigma_{t_n,25\Delta,\text{rr}}, \sigma_{t_n,25\Delta,\text{str}} \right\}$.
Let us also write \( x^{s_1} = (s_1, \theta^{s_1}) \).

**Step 4:**

Construct a random variable \( \Theta^{s_1} \) such that

\[
E_{t_0} [C(X^{s_1}, K, T)] = C(x^{s_1}, K, T),
\]

where \( X^{s_1} = (s_1, \Theta^{s_1}) \).

**Step 5:**

Sample \( \theta_1 = \Theta^{s_1}(\omega) \) according to a distribution constructed in Step 4.

As a result, we have \( x_1 = \{s_1, \theta_1\} = X_1(\omega) \). Now change notations so that \( x_1 = x_0 \) and start from Step 1. Repeat this procedure until \( t_e \) is reached and evaluate \( P(w) = P(s_{t_e}, A(x_{t_e})) \). **End of the algorithm.**

Step 4, as the least trivial one, deserves a special explanation. There are several different ways to construct \( \Theta^{s_1} \) such that (43) is satisfied. Here is one, perhaps the simplest and most naive, method.

First, let us focus our attention one particular tenor \( T \in \{T_1, \ldots, T_n\} \). The other tenors are dealt with similarly. Note that the volatility quotes for \( T \) correspond to prices of three standard vanilla option strategies (for brevity drop tenor \( T \) from the notation):

\[
\{\sigma^{\text{atm}}_{s_1}, \sigma^{\text{rr}}_{s_1}, \sigma^{\text{str}}_{s_1}\} \rightarrow \{V^{\text{atm}}_{s_1}, V^{\text{rr}}_{s_1}, V^{\text{str}}_{s_1}\} \tag{44}
\]

Here \( V^{\text{atm}}_{s_1} \) is a \( t_1 \)-price of an at-the-money option, implied by \( \sigma^{\text{atm}}_{s_1} \). Similarly, \( \sigma^{\text{rr}}_{s_1} \) and \( \sigma^{\text{str}}_{s_1} \) imply the prices of a 25-delta risk reversal (\( V^{\text{rr}}_{s_1,25\Delta,rr} \) and \( V^{\text{str}}_{s_1,25\Delta,str} \)) respectively. We will not discuss the exact definitions of these vanilla strategies. Let us assume that for any European option with the same maturity, its price is a linear function of these strategies’ prices:

\[
V_K = a_K V^{\text{atm}}_{s_1} + b_K V^{\text{rr}}_{s_1,25\Delta,rr} + c_K V^{\text{str}}_{s_1,25\Delta,str} \tag{45}
\]

We have dropped all the time-related notation, but introduced a subscript \( K \) to denote that this equation holds for every strike (with weights being strike dependent). This equation is a rather reasonable approximation of reality, often used by the FX traders (see, for example, a well-known reference on FX option pricing models, Lipton[18]).

It is easy to see that when (45) holds, equation (43) is guaranteed for all options if it is guaranteed to the standard strategies introduced above (by the
linearity of the expectation operator). Hence, in order to ensure (43) we can introduce $\Theta^{s_1}$ through the following:

$$V^{s_1}_{t_1, atm}(\omega) : \Omega \rightarrow \{V^{s_1}_{t_1, atm} + \delta_{atm}, V^{s_1}_{t_1, atm} - \delta_{atm}\}, \quad (46)$$

where the higher and lower values are assumed with equal probability.

Similarly,

$$V^{s_1}_{t_1, 25\Delta, rr}(\omega) : \Omega \rightarrow \{V^{s_1}_{t_1, 25\Delta, rr} + \delta_{rr}, V^{s_1}_{t_1, 25\Delta, rr} - \delta_{rr}\}, \quad (47)$$

$$V^{s_1}_{t_1, 25\Delta, str}(\omega) : \Omega \rightarrow \{V^{s_1}_{t_1, 25\Delta, str} + \delta_{str}, V^{s_1}_{t_1, 25\Delta, str} - \delta_{str}\}. \quad (48)$$

These definitions imply the distribution of $\Theta^{s_1}$ in an obvious way.

Additional care has to be taken in order to ensure that $x_i = \{(s_i, \theta^{s_1})\}$ is always admissible. This can be guaranteed when choosing $\delta_{atm}, \delta_{rr}, \delta_{str}$ on each of the simulation steps. For example, every time when a statically-arbitrageable surface is approached, $\delta_{atm}, \delta_{rr}, \delta_{str}$ (or one of them) can be halved.

6 Conclusions

The problem of capturing the stochastic nature of smile surfaces in mature option markets is of great importance, affecting both pricing and risk management practice. As reflected in a number of both empirical and theoretical works ([1],[8],[24],[23] and others), there is a clear need for a market model of stochastic implied volatility surfaces. In this paper, we have presented a modelling approach a that allows no-arbitrage market models to be created.

The immediate practical application of this approach is a simulation procedure for the joint evolution of spot and option prices. Building such a simulation and studying how well it will capture the real-life market behavior is the next direction of our research. Studying the impact of a stochastic smile model on prices of heavily exotic options (in particular, compound options) is another possible by-product of this research direction.

It also remains to see whether the theoretical framework developed here can be extended to a continuous time case.

References


