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An alternative proof of the
nowhere-zero 6-flow theorem

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Abstract

The nowhere-zero 6-flow theorem of Seymour is proven by construction.

Key words: bridgeless graph, nowhere-zero 6-flow, construction.

AMS Subject Classification: 05C38, 05C99.

1 Introduction

We consider connected cubic bridgeless graphs. For terminology we refer to the book of Holton and Sheehan [1].

Definition 1.1 *A graph G is a cycle dominated graph if G has a cycle C such that $G \setminus VC$ is acyclic.*

Notice that if a graph G has a dominating cycle, then G is a cycle dominated graph.

Lemma 1.1 *Every cubic hamiltonian graph has a nowhere-zero 6-flow with odd flow value on each edge of its hamiltonian cycle.*

Proof The number of vertices of a cubic hamiltonian graph is even. We can colour the edges in its hamiltonian cycle by two colours x and y alternatingly.

Colour the remaining edges by the colour z . Send a flow of value 1 through xz -cycles and a flow of value 3 through yz -cycles in arbitrary direction. Since each edge in the cycle is covered once, the flow value on each edge of the cycle is 1 or 3. ■

Lemma 1.2 *If a cubic bridgeless graph G has a cycle C such that $G \setminus VC$ is acyclic, then G has a nowhere-zero 6-flow.*

Proof Let us consider a cycle dominated graph G . There is a cycle C such that $G \setminus VC$ is a forest. Since a forest is 2-vertex colourable, we can orient the edges of G such that each vertex in $VG \setminus VC$ becomes a sink or source, see Figure 1.

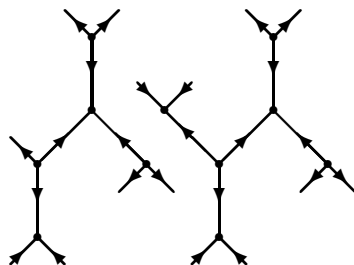


Figure 1: Sinks and sources in a forest.

Send a flow of value 2 through each edge that is incident with a vertex in $VG \setminus VC$ according to the orientation. Total inflow and total outflow at each vertex are equal to 0 modulo 6.

We can find a 6-flow f_1 on G such that each edge incident with a vertex in $VG \setminus VC$ has a flow of value 2, each edge of C has a flow of value 0, 2 or 4, and each remaining edge has a flow of value 0. This flow f_1 is constructed by dealing with the forest $G \setminus VC$ in the way described. The vertices of C adjacent to a vertex of $VG \setminus VC$ divide C into parts. Choose one of these parts and send a flow of value 0 in an arbitrary direction, i.e. simply choose flow value 0 on this part. The flow values on the other parts are now determined. This construction is consistent since the sum of the flows at vertices of $G \setminus VC$ is equal to 0 modulo 6 and so is the sum of inflow and outflow values from and to these vertices for vertices of C . We want all orientations to be chosen towards the cycle C for edges incident to

vertices of C . This means that we have to change flow values from 2 to 4 when the orientation is to be changed. Call these flow values $a_i, i = 1, 2, \dots, p$. a_1 and a_p are inflow values for the vertices of C determining the part that is chosen to get a flow of value 0. By the chosen orientations the flow values must be $\sum_{i=1}^k a_i, k = 1, 2, \dots, p$ on the edges of the cycle C . In particular the flow on the part between a_p and a_1 must be $\sum_{i=1}^p a_i$, but we argued that this is equal to 0 modulo 6.

Consider $G \langle VC \rangle$, the subgraph of G induced by VC . Since C is a cycle, it is clear that $G \langle VC \rangle$ is a hamiltonian graph. Note that $G \langle VC \rangle$ is not cubic anymore. Let C^* be the hamiltonian graph derived from $G \langle VC \rangle$ by contracting edges incident with vertices of degree 2. By Lemma 1.1, we can find a nowhere-zero 6-flow f_2^* with odd flow value on each edge of its cycle. For an example, see Figure 2.

Since $G \langle VC \rangle$ is a subdivision of C^* , $G \langle VC \rangle$ also has a nowhere-zero 6-flow f_2 with odd flow value on each edge of its cycle. Since $G \langle VC \rangle$ is a subgraph of G , f_2 can be used to construct a 6-flow on G by putting a flow of value 0 on each edge that is incident with a vertex in $VG \setminus VC$. It is clear that $f_1 + f_2$ is a nowhere-zero 6-flow on the graph G . Notice that each edge in $EG \setminus EC$ has a flow of value 2 or 4 and each edge in EC has a flow of value 1, 3 or 5. ■

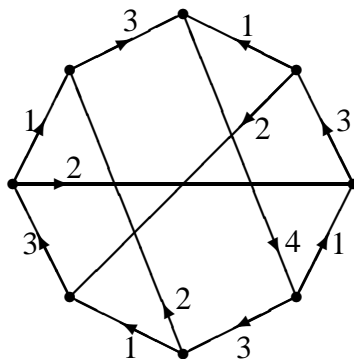


Figure 2: A nowhere-zero 6-flow f_2^* on a hamiltonian graph C^* derived from $G \langle VC \rangle$ by contracting edges incident with vertices of degree 2.

2 Nowhere-zero 6-flows

Let G be a bridgeless graph. A family \mathcal{C} of cycles $\{C_1, C_2, \dots, C_s\}$ of G is a *spanning cycle* of G if $|VG| = |VC| \equiv |\bigcup_{i=1}^s VC_i|$. Note that $EC = \bigcup_{i=1}^s EC_i$.

A cubic graph G is a cycle dominated graph if G has a spanning cycle consisting of a single cycle. It has been proven in Lemma 1.2 that a cycle dominated graph has a nowhere-zero 6-flow. The important point in the proof of this lemma is that we can colour the vertices $G \setminus VC$ by two colours and make them sinks and sources.

Definition 2.1 A cubic graph G is called an *almost cycle dominated graph* if G has a spanning cycle $\{C_m, C_1, C_2, \dots, C_s\}$ such that if we contract C_1, C_2, \dots, C_s and remove C_m , the rest is a forest.

Lemma 2.1 An almost cycle dominated graph has a nowhere-zero 6-flow.

Proof Suppose $\mathcal{C} = \{C_m, C_1, C_2, \dots, C_s\}$ is a spanning cycle of G such that if we contract C_1, C_2, \dots, C_s and remove C_m , the rest is a forest. Remove all chords of the cycles of \mathcal{C} . If we remove C_m and one edge of each other cycle of \mathcal{C} , then the rest is a forest. By the construction of a 6-flow f_1 as in the proof of Lemma 1.2, we can find a 6-flow f_{11} of G such that each chord of a cycle of \mathcal{C} has a flow of value 0, each edge in a cycle of \mathcal{C} has a flow of value 0, 2 or 4 modulo 6 and each remaining edge has a flow of value 2.

Again, by the construction of a 6-flow f_2 as in the proof of Lemma 1.2, we can find a 6-flow f_{22} of G such that each chord of a cycle of \mathcal{C} has a flow of value 2 and an odd flow value on each edge of a cycle of \mathcal{C} . It is clear that $f_{11} + f_{22}$ is a nowhere-zero 6-flow on the graph G . Notice that each edge of a cycle of \mathcal{C} has a flow of value 1, 3 or 5 and the other edges have a flow of value 2. ■

Let A be a configuration, as in Figure 3.a, that is contained in a cubic graph. Define a configuration B by removing the edges (v_1, v_2) and (v_3, v_4) and adding edges (v_1, v_4) and (v_2, v_3) , See Figure 3.b. This transformation we call a *shift* from A to B .

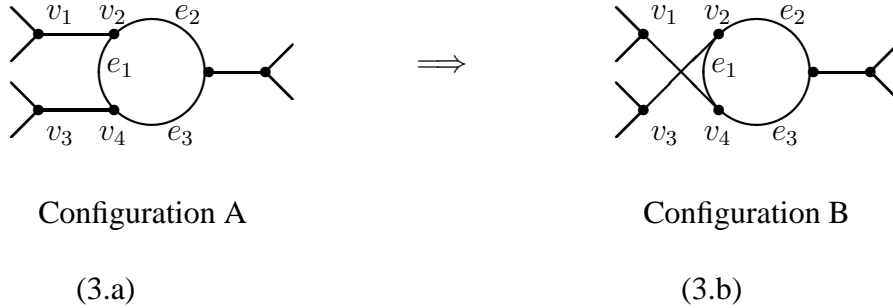


Figure 3: A shift.

Lemma 2.2 Consider a graph G that contains configuration A. Define a graph G^* by a shift from A to B. If G has a nowhere-zero 6-flow such that (v_1, v_2) and (v_3, v_4) have a flow of value 2 and e_2 has an odd flow value in arbitrary direction, then G^* has a nowhere-zero 6-flow such that (v_1, v_4) and (v_3, v_2) have a flow of value 2 in the same direction as (v_1, v_2) and (v_3, v_4) and e_1 still has an odd flow value.

Proof If in G (v_1, v_2) and (v_3, v_4) have a flow of value 2 in the same direction, then nothing is changed. If in G the flows of value 2 in (v_1, v_2) and (v_3, v_4) are not in the same direction, then to the flow value of e_1 4 is added from v_2 to v_4 .

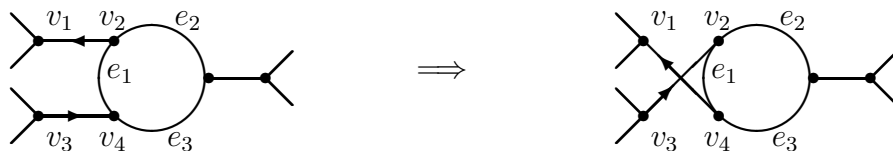


Figure 4: Flow of value 2 in (v_1, v_2) and (v_3, v_4) not in the same direction. ■

A consequence of Lemma 2.2 is that, for our nowhere-zero 6-flow problem, the position of vertices on a cycle of \mathcal{C} is not important.

Our main goal is to prove Theorem 2.1, the 6-flow theorem, by construction. We will give an outline of the proof by dealing with an example.

Consider a cubic graph G and its spanning cycle $\mathcal{C} = \{C_m, C_1, C_2, C_3, C_4\}$. See Figure 5.

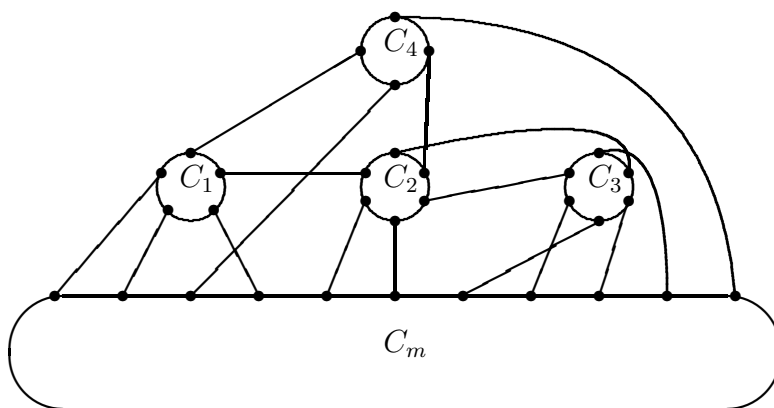


Figure 5: General structure of a cubic graph G and its spanning cycle $\mathcal{C} = \{C_m, C_1, C_2, C_3, C_4\}$.

G is not an almost cycle dominated graph since after we contract C_1, C_2, C_3, C_4 still a cycle exists even when removing C_m , see Figure 6. Cycles (C_1, C_2, C_4, C_1) and (C_2, C_3, C_2) are created.

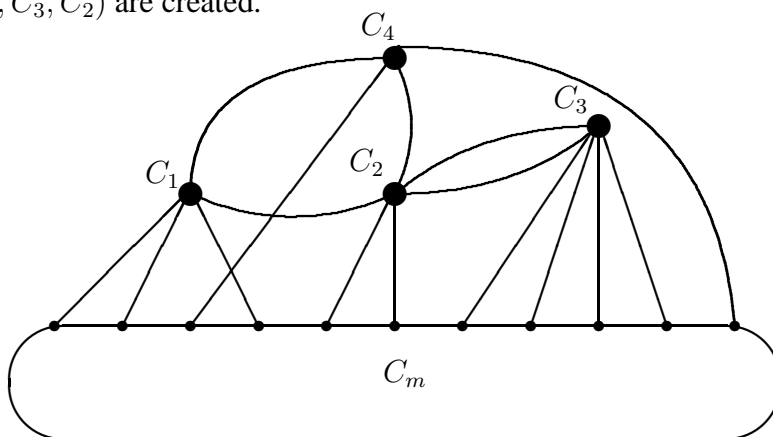


Figure 6: A typical structure after contracting cycles.

If in G we remove arbitrary edges that correspond to an edge of cycle (C_1, C_2, C_4, C_1) and an edge of cycle (C_2, C_3, C_2) , we get, after contracting edges of degree 2, an almost cycle dominated graph G^* . See Figure 7. This procedure to get an almost cycle dominated graph can be applied in general.

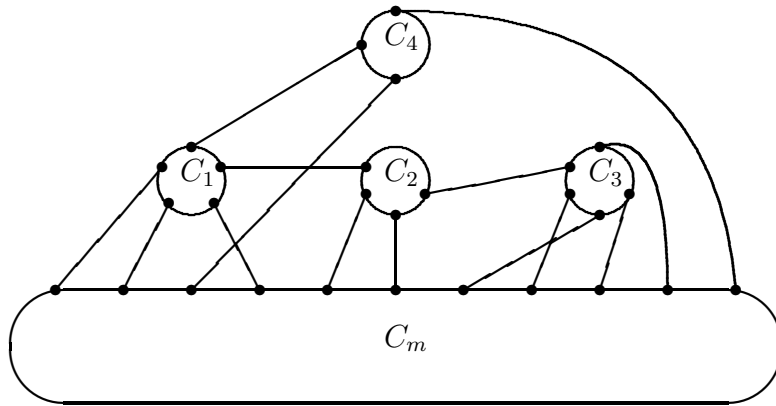


Figure 7: An almost cycle dominated graph G^* .

By construction of a nowhere-zero 6-flow as in Lemma 2.1, a nowhere-zero 6-flow f_1 of G^* can be constructed the directions of which can be seen in Figure 8. An arrow on an edge indicates a flow of value 2. The other edges, of C_m , C_1 , C_2 , C_3 or C_4 , have an odd flow value.

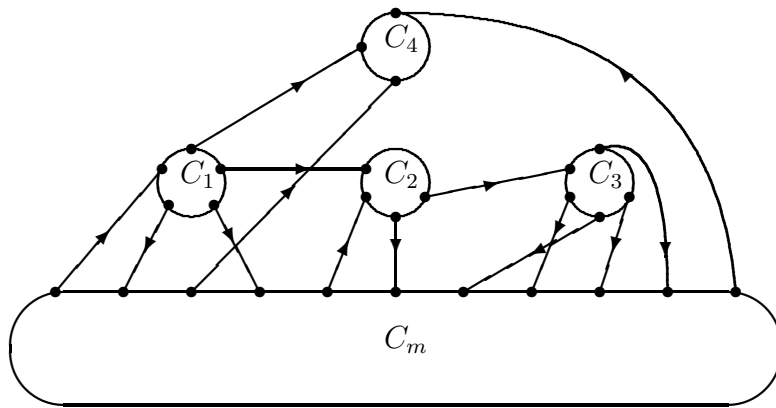


Figure 8: A nowhere-zero 6-flow f_1 on an almost cycle dominated graph G^* .

Consider Figure 9. To get a nowhere-zero 6-flow of G , we have to send a flow of value 2 around a cycle that contains edge (v_5, v_6) and around a cycle that contains (v_8, v_9) . Note that by sending a flow of value 2, the flow value of each edge in the cycles C_m, C_1, C_2, C_3 or C_4 is still odd.

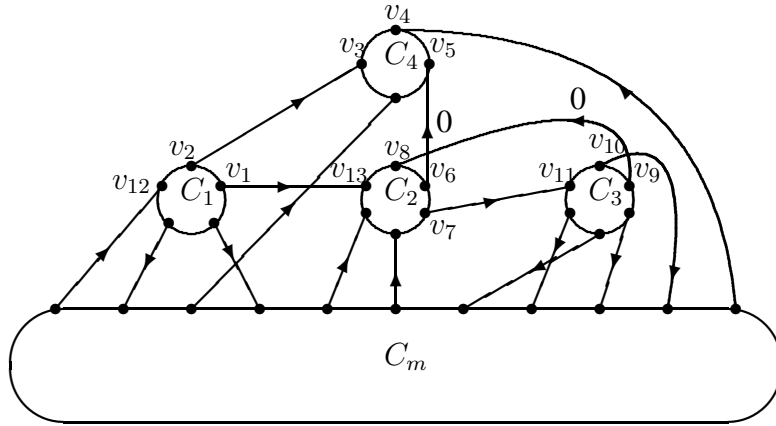


Figure 9: A 6-flow of G .

By sending a flow of value 2 around cycle $(v_7, v_{11}, v_{10}, v_9, v_8, v_6, v_7)$, a flow of value 2 on edge (v_7, v_{11}) is changing to 4 or is still 2, in the opposite direction, and the flow of value 0 on edge (v_9, v_8) is changing to 2.

We cannot send a flow of value 2 around cycle $(v_6, v_5, v_4, v_3, v_2, v_1, v_{13}, v_8, v_6)$ since a flow of value 0 on (v_2, v_3) is created.

We can solve this problem by shifting the positions of v_2 and v_{12} before we use the construction method of Lemma 2.1. After we get a nowhere-zero 6-flow for the resulting graph, we can shift the positions back. The result, f_2 , can be seen in Figure 10. Note that various flow directions have changed.

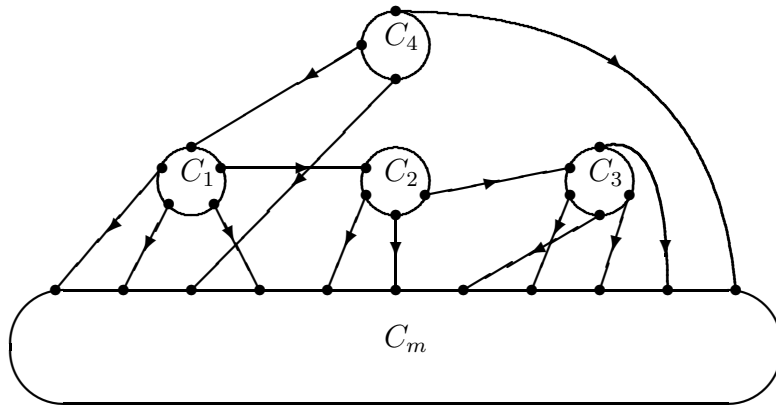


Figure 10: A nowhere-zero 6-flow f_2 on an almost cycle dominated graph G^* .

We finally consider Figure 11. Note that, in particular, the direction of the flow on (v_3, v_2) has changed. From the nowhere-zero 6-flow of the almost cycle dominated graph G^* , we can now find a nowhere-zero 6-flow of G by sending a flow of value 2 around cycle $(v_6, v_5, v_4, v_3, v_2, v_1, v_{13}, v_8, v_6)$ and also a flow of value 2 around cycle $(v_7, v_{11}, v_{10}, v_9, v_8, v_6, v_7)$, to obtain a nowhere-zero 6-flow for the original graph.

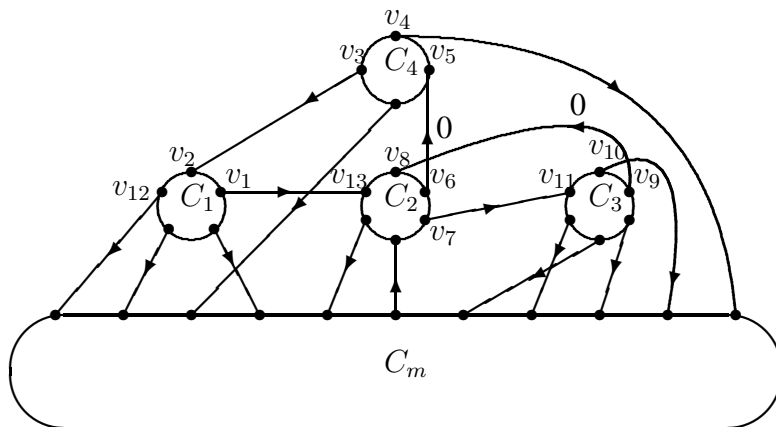


Figure 11: A 6-flow on G .

Theorem 2.1 (Seymour [2]) *A bridgeless cubic graph has a nowhere-zero 6-flow.*

Proof The construction steps carried out in the example can be carried out in the general case as well.

There are seven steps in the construction of the nowhere-zero 6-flow :

1. Finding a spanning cycle $\mathcal{C} = \{C_1, C_2, C_3, \dots, C_n\}$.
2. Removing all chords of each cycle of \mathcal{C} .
3. Choosing C_m , a cycle of \mathcal{C} , and contracting the other cycles.
4. Removing C_m . If there is a cycle C_k , remove an edge of this cycle. Note that the rest is a forest.
5. Applying the construction of a 6-flow of Lemma 2.1. By the shifting method, the direction of flow values on the edges of C_k can be forced to be in the same direction.
6. Sending a flow of value 2 around the C_k in the same direction of the direction of flow value 2 of the edges of C_k .
7. Applying the construction of Lemma 1.1, so that all chords have a flow of value 2.

■

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- [2] P. Seymour. Sums of circuits. In J. Bondy and U. S. R. Murty, editors, *Graph Theory and Related Topics*, pages 431–355. Academic Press, New York, 1979.