DOMAIN-INDEPENDENT THERMOELASTIC COUPLING SUITABLE FOR AERO-THERMOELASTIC MODELING

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In this paper, a novel domain-independent nonlinear thermoelastic model suitable for aero-thermoelastic investigations is generated with which a continuous power transformation frame between the elastic and thermal domains is formed. To achieve this, first, a distinctive power distribution of each domain is defined by means of the Bond graph notation, and the corresponding governing equation for each domain is extracted on the basis of the port-based approach. Next, to provide a continuous power transmission between the elastic and thermal domains, a reversible coupling is designed and the corresponding generalized equation is derived that satisfies the Maxwell reciprocity. Finally, by employing the generated coupling, the elastic and thermal domains are connected dynamically. Implementing the energy-based strategies, a physical model that is capable of dynamically capturing the reversible interactions between the thermal and elastic domains while preserving the fundamental physical natures of the thermoelastic phenomena, is generated. The generated model is domain-independent and, in principle, more suited to be connected to other physical domains than conventional counterparts. This ability of the generated model provides a unique benefit to the development of appropriate schemes for controlling structural vibrations under aerothermal loads.

1. Introduction

From experiment it is evident that deformation of a structure is often associated with temperature change. A time-varying mechanical loading onto a structure causes not only a varying strain of the structure, but also a temperature distribution within the structure that is changing with time. Conversely, heating a structure produces not only temperature change in the structure, but also deformation of the structure [1]. Lack of attention to this thermoelastic behavior in aerodynamic systems, where aero-thermoelastic phenomena are dominant, may result in severe interruptions in the desired performances [2]. This fact highlights a critical need to investigate the thermoelastic behavior for such systems for a better control outcome.

To investigate the thermoelastic behavior, a model that can capture the nonlinear nature of thermoelasticity would be most desirable. Since the thermoelastic behavior is a multi-physical domain phenomenon, the desired model must not only be able to capture the dynamic interactions between the elastic and thermal domains, but also be compatible with other physical domains (such as aero-thermo domains) that may also be involved in the behavior. A physical and domain-independent modelling strategy is thus preferable.

The aim of this paper is to generate a domain-independent, nonlinear, lumped-parameter, physical model of thermoelasticity for use in multi-physical system analysis and control investigations. Using the Bond graph (BG) techniques [3-6], a reversible coupled thermoelastic model that can provide the dynamic insights into the relations between the variables and components of the system and describe
the thermoelastic behavior of the system, is generated. In contrast to other comparable counterparts [7-10], the suggested model offers a direct physical meaning to the coupled thermoelastic behavior, while providing a good level of accuracy. The novelty of this modelling approach lies in its ability to abolish widely-used conventional assumptions through the use of a port-based approach [11-13]. The generated model can elegantly illustrate the thermoelastic behavior under aero-thermo loads, and provides a low time-cost platform based on which simulation of aero-thermoelastic control strategies can be developed.

2. Reversible thermoelastic BG coupling

According to the port-based approach, the reversible coupling of multi-physical systems can be represented by a multiport energy storage element. In this section we derive the generalized equation of a thermoelastic 2-port energy storage (ℂ) shown in Fig. 1. This multiport energy storage in principle forms the reversible connection between the elastic and thermal domains with which the internal continuous power transmission between the two domains is possible. We name ℂ as reversible thermoelastic energy storage. Implementing this element, we anticipate to capture the thermal dynamic changes via alteration in the elastic domain and vice versa, in an energy conservative manner.

![Figure1 : Two-port C model of the thermoelastic energy storage.](image)

In the suggested coupling, contrary to the conventional methods, entropy rate is considered as the time derivative of the state for the thermal domain instead of heat, in order to make the model compatible with other physical domains. To define the constitutive equations of the potentials (F, T) for the suggested ℂ as integral causality is preferred, the true energy is considered as a function \( U = U(q_m, S) \), where \( q_m \) and \( S \) are the deformation and entropy of the system, respectively. Accordingly:

\[
dU = \frac{\partial U}{\partial q_m} dq_m + \frac{\partial U}{\partial S} dS
\]

By definition:

\[
\left( \frac{\partial U}{\partial q_m} \right)_S = F, \quad \left( \frac{\partial U}{\partial S} \right)_{q_m} = T
\]

\( F = F(q_m, S), \quad T = T(q_m, S) \)

To define the conjugate potentials (efforts) for the ℂ storage, let’s start with the temperature, \( T \):

\[
dT = \left( \frac{\partial T}{\partial q_m} \right)_S dq_m + \left( \frac{\partial T}{\partial S} \right)_{q_m} dS
\]

According to the thermal energy at constant volume:

\[
dU_{\text{thermat}} = C_v dT
\]

\[
\left( \frac{\partial T}{\partial S} \right)_{q_m} = \frac{T}{C_v}
\]

where \( C_v \) is the specific heat in constant volume. To find the first term of Eq. (4), consider the Maxwell reciprocity and the 1D Hook’s law:

\[
\left( \frac{\partial T}{\partial q_m} \right)_S = \left( \frac{\partial F}{\partial S} \right)_{q_m}
\]

\[
F = AE \frac{q_m}{L} + \alpha AE (T - T_0)
\]

\[
\left( \frac{\partial T}{\partial q_m} \right)_{q_m} = \alpha AE \left( \frac{\partial T}{\partial S} \right)_{q_m} = \alpha AE \frac{T}{C_v}
\]

where \( A, E, L, \alpha \) and \( T_0 \) are the element section area, material stiffness, element’s length and axial heat expansion coefficient, and reference temperature, respectively. Substituting Eqs. (6) and (9) into Eq. (4) yields:
\[
\frac{dT}{T} = \frac{\alpha AE}{C_v} dq_m + \frac{1}{C_v} dS
\]  \hspace{1cm} (10)

Considering an unstressed element, and assuming constant specific heat, by integrating Eq. (10) the thermal effort can be presented as:

\[
T = T_0 e^{\frac{\alpha AE}{C_v} dq_m e^{\frac{(S-S_0)}{C_v}}} \hspace{1cm} (11)
\]

Using Eq. (11) to change the causality of Eq. (8), the elastic effort, \( F \), is formed as:

\[
F = AE \frac{q_m}{L} + \alpha AET_0 (e^{\frac{\alpha AE}{C_v} dq_m e^{\frac{(S-S_0)}{C_v}}} - 1) \hspace{1cm} (12)
\]

Substituting the two constitutive relations in Eqs. (11) and (12) into Eq. (1) forms the true energy function of the presented storage with respect to the considered independent state variables:

\[
dU = \left( AE \frac{q_m}{L} + \alpha AET_0 (e^{\frac{\alpha AE}{C_v} dq_m e^{\frac{(S-S_0)}{C_v}}} - 1) dq_m + \left( T_0 e^{\frac{\alpha AE}{C_v} dq_m e^{\frac{(S-S_0)}{C_v}}} - \frac{\alpha AE}{C_v} dq_m \right) dS \right) \hspace{1cm} (13)
\]

The resulting nonlinear multiport energy function of the thermoelastic domain, given by Eq. (14), contains a contribution relating to the displacement/strain and a contribution relating to the entropy, thus collectively showing the combined effect of thermoelasticity. It is clear that the contributions of the thermal and elastic domains can be individually expressed via the extensive variables (displacement and entropy) of each of these two domains. This feature makes the proposed modeling technique domain-independent and physical. Knowing that the dynamic coupling between the thermal and elastic domains can be represented by the obtained \( \mathbb{C} \) element, the next step is to install the suggested storage element in a thermoelastic junction structure.

### 3. 1D thermo-elastic BG model

Given that the focus of this paper is to preserve the physical insight of the thermoelastic behavior while modelling it, a thermoelastic BG model containing separate energy distribution schemes for each of the involving domains, is proposed. To achieve this, a domain-independent elastic model presented in [14] will be connected to a compatible thermal model presented in [15]. The key connection joint here is the generated \( \mathbb{C} \) which describes the reversible, energy conservative, nonlinear dynamic coupling between the elastic and thermal domains.

![Thermoelastic BG model](image)

The 1D reticulated geometry shown in Fig. 2 (a) is used to describe the propagation of energy within the elastic domain. In Fig. 2 (c), the top energy line shows the BG representation of the chosen elastic domain [14]. It is seen that each element is composed of two storage components, generalized capacitor (spring) and generalized inertia (mass). On the basis of the considered arrangement, the total internal energy can be stored at the boundaries in the form of kinetic energy and inside each
element in the form of elastic energy. Correspondingly, $p_i$ and $q_i$ denote, respectively, the momentum of each boundary and the relevant deformation of each element.

To form the energy distribution in the thermal domain, the 1D compatible conduction reticulation shown in Fig. 2 (b) is considered. The corresponding BG representation is given by the bottom line of Fig. 2 (c). It is shown that each thermal element is considered as a C-type storage element with the state variable, $q_{Ti}$, denoting the amount of stored entropy $S_i$. In the selected conductive model, each $C$ element is connected to its adjacent elements via the junction elements containing dissipative characteristics [15].

To form the coupled thermoelastic model, the chosen thermal BG model is connected to the elastic BG model by replacing the capacitor of each domain with the generated storage element. The products of the effort and flow of each side of the 2-port storage element ($F_i$, $q_{mi}$, and $T_i$) then represent the conservative and reversible power interchange between the elastic and thermal domains. Considering the generated coupling, and by employing the constitutive Eqs. (11) and (12) for the coupled field, the governing equations for a single, port-based thermoelastic element ($i_{th}$ element in a chain) can be extracted as:

$$\dot{p}_j = A_i E_i \frac{q_{m_{i-1}}}{T_{0j}} - A_{i+1} E_{i+1} \frac{q_{m_{i+1}}}{T_{0j}} + \alpha_i A_i E_i T_{0j} \left( \frac{\alpha_i A_i E_i G_m}{c_{v_i} \alpha_i e^{\frac{(s_i - s_{0i})}{c_{v_i}}} - 1} \right) - \alpha_{i+1} A_{i+1} E_{i+1} T_{0j+1} \left( \frac{\alpha_{i+1} A_{i+1} E_{i+1} G_{m_{i+1}}}{c_{v_{i+1}} \alpha_{i+1} e^{\frac{(s_{i+1} - s_{0i+1})}{c_{v_{i+1}}}}} - 1 \right)$$

(14)

$$\dot{q}_{mi} = \frac{p_{j-1}}{I_{j-1}} \frac{p_j}{I_j}$$

(15)

$$\dot{S}_{Ti} = \frac{1}{R_{j-1}} T_{0j-1} \left( \frac{a_{i+1} A_{i+1} E_{i+1} G_{m_{i+1}}}{c_{v_{i+1}}} e^{\frac{(s_{i+1} - s_{0i+1})}{c_{v_{i+1}}}} \right) - T_{0j} e^{\frac{a_{i+1} A_{i+1} E_{i+1} G_{m_{i+1}}}{c_{v_{i+1}}}} \left( \frac{(s_{i+1} - s_{0i+1})}{c_{v_{i+1}}} \right) - \frac{1}{R_j} T_{0j} e^{\frac{a_{i} A_{i} E_{i} G_{m_{i}}}{c_{v_{i}}}} \left( \frac{(s_{i} - s_{0i})}{c_{v_{i}}} \right)$$

(16)

$$\dot{S}_{gen}^i = \frac{a_{i} A_{i} E_{i} G_{m_{i}}}{c_{v_{i}}} e^{\frac{(s_{i} - s_{0i})}{c_{v_{i}}}} \left( T_{0i+1} - T_{0i} \right) - \frac{a_{i+1} A_{i+1} E_{i+1} G_{m_{i+1}}}{c_{v_{i+1}}} \left( \frac{(s_{i+1} - s_{0i+1})}{c_{v_{i+1}}} \right)$$

(17)

$$I_j = \frac{m_{i+1} + 2}{2} + \frac{m_{i} - 1}{2}$$

(18)

$$R_j = \frac{1}{2} \left( \frac{a_{i} A_{i} E_{i} G_{m_{i}}}{k_{i+1} A_{i+1}} e^{\frac{(s_{i} - s_{0i})}{c_{v_{i}}}} + \frac{a_{i+1} A_{i+1} E_{i+1} G_{m_{i+1}}}{k_{i+1} A_{i+1}} e^{\frac{(s_{i+1} - s_{0i+1})}{c_{v_{i+1}}}} \right)$$

(19)

where $I_j$ is the boundary inertia and $R_j$ is the thermal resistance parameter of each junction element. Eqs. (15) and (16) represent, respectively, the rate of the element’s boundary momentum and deformation, and Eq. (17) denotes the element’s entropy rate as a nonlinear function of the considered extensive states ($q_{m_{i}}$ and $S$) as well as the geometrical and material parameters. It is clear that the thermal state appears in the elastic domain’s momentum equation, and the elastic state appears in the thermal domain’s entropy equation. This in principle indicates the influence of the thermal domain on the elastic domain and vice versa. Eq. (18) demonstrates the amount of generated entropy occurring in the thermal domain. Clearly, this equation satisfies the second thermodynamic law, as the amount of the generated entropy is always greater than zero. Further attentions to the generated state equations reveal that, although the elastic domain is considered to be non-dissipative (pure elastic), the irreversibility of the thermal domain can affect the elastic domain’s behavior via the reversible connectivity of these two domains. This occurs as the dynamics of the elastic domain are under the influence of the thermal domain, while the dynamics of the thermal domain are bundled with the existing irreversibility.
The above explanation demonstrates the state of the art implementation of the BG approach in unveiling the meaningful physical insight of the thermoelastic phenomena in the overall internal dynamics of the multi-physical systems. It should be mentioned that to obtain a complete coupling between these two domains, the energy dissipation in the elastic domain also needs to be taken into account. This is beyond the scope of the current study.

4. Simulation and analysis

To evaluate the ability of the generated BG model to capture the thermoelastic phenomena, a simple beam axial vibration is simulated under thermoelastic loading. The geometrical and material parameters of the beam are given in Table 1, and the 1D axial thermomechanical behavior of the beam is to be investigated. To generate the discretised geometry, the chosen beam is reticulated into 10 uniform elements with the first and last elements being the boundary elements that receive external mechanical and thermal loads. It is assumed that the side surface of the beam is fully isolated and the beam is stress-free initially in the ambient room temperature. Accordingly, the internal thermal and elastic behaviours of the system, and the corresponding energetic behaviour of each domain is presented in Figs. 4 to 6 with respect to the thermal and mechanical loads presented in Fig. 3.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description (unit)</th>
<th>Value</th>
<th>Symbol</th>
<th>Description (unit)</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>Conduction coefficient (( J/m.K ))</td>
<td>( 2.73e^2 )</td>
<td>( m )</td>
<td>Beam mass (( Kg ))</td>
<td>( 5.67e^{-2} )</td>
</tr>
<tr>
<td>( E )</td>
<td>Young modulus (( J/m^3 ))</td>
<td>( 6.9e^{10} )</td>
<td>( A )</td>
<td>Cross section area (( m^2 ))</td>
<td>( 1e^{-4} )</td>
</tr>
<tr>
<td>( c_p )</td>
<td>Specific heat (( J/Kg.K ))</td>
<td>( 8.97e^2 )</td>
<td>( l )</td>
<td>Length (m)</td>
<td>( 2.1e^{-1} )</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>Linear expansion (1/K)</td>
<td>( 2.22e^{-5} )</td>
<td>( M )</td>
<td>Molar mass (( kg/mol ))</td>
<td>( 2.698e^{-2} )</td>
</tr>
<tr>
<td>( S_0 )</td>
<td>Reference Entropy (( J/Kg.K ))</td>
<td>( 2.83e^1 )</td>
<td>( n )</td>
<td>Number of segments</td>
<td>10</td>
</tr>
</tbody>
</table>

To check the ability of the suggested thermoelastic BG model to track the internal dynamics under complex loading conditions, a mechanical load accompanied by a separate thermal load, shown respectively in Fig. 3 (a) and (b), are applied to both ends of the beam symmetrically. These loads are considered as flow sources into the system.

The resultant elements’ deformations and temperature changes are depicted in Fig. 4 (a) and (b), respectively. Only the information of half of the beam elements is presented, as the loading of the structure is chosen to be symmetrical. The contraction of each element is considered to gain positive value. It is seen that the impacts of the thermal source are dominant in forming the deformations of
the side elements, while the center elements follow the pattern of a mixed loading situations. This behavior in fact highlights the slow conductive dynamics of the system in comparison with the elastic dynamics. It also indicates that under thermoelastic loading, a mixed expansion-contraction pattern of deformations, with respect to the position of the thermal source, can occur within the system. Neglecting this phenomenon in a controlled structure may result in deteriorations of the control system performance. For instance, inside a noise-control structure, this phenomenon may result in misleading control signals.

Fig. 5 demonstrates the corresponding energetic behavior of the beam for a longer period. Fig. 5 (a) indicates the stored energy in the elastic domain, and Fig. 5 (b) represents the corresponding energy change in the thermal domain. As can be seen, the style of the energy consumption of these two domains is totally different from each other. In the elastic domain the potential parameter of the domain (Force) tends to have a non-dissipative and conservative behavior in each cycle, whereas in the thermal domain an accumulative-dissipative pattern can be observed. To explain the shifting pattern of the obtained results, one can consider that the added amount of energy to the system as a result of the thermal domain irreversibility can increase the temperature and, consequently, the internal tension of the system. This unwanted energetic lift inside the system in long term may result in unwelcoming deformations of the system. For a position-controlled structure, this phenomenon may alter the energy consumption of the control system.
The fact revealed in Fig. 5 indeed demonstrates the benefit of implementing conservation of energy in the suggested thermoelastic BG model. The clock-wise rotation in the thermal domain and the counter-clock-wise rotation in the elastic domain show that the amount of energy transferred between these two domains is always equal opposite to each other. This, thanks to the generated reversible energy storage, means that energy exiting from one domain is interred into the other domain without missing. The different energetic patterns between the thermal and elastic domains, although indicate a weak coupling of the elastic domain with the thermal domain, do show how this weak interaction can alter the performance of the system.

Having a clear understanding of the energetic behavior of a multi-physical system, especially in the form of a separate graph for each of the involving domains, can be a supportive tool for managing the energy consumption of the entire system. For example, knowing the local thermal behavior of the system about the natural frequency, e.g., shown in Fig. 6 (a), in order to stabilize the system one can exclude a certain amount of entropy from the system via locally cooling the system. Alternatively, as in the case of a noise reduction application, knowing the local elastic behavior of the system, e.g., shown in Fig. 6 (b), in order to suppress the undesired deformation one can compensate the impact of the stress growth inside the system via, for instance, implementing an effective active control mechanism.

5. Conclusion

In this study, by means of the BG method, a domain-independent, nonlinear, coupled thermoelastic model is generated for investigating the thermomechanical behavior of a multi-physical system. In this model, via the suggested energy conservative coupling, the reversible dynamic interactions between the elastic and thermal domains are considered. The generated model is capable of not only describing the dynamic behavior of the system, but also providing a useful power frame within which the energy distribution of the system with respect to each of the involved domains is distinguishable.

The rational compatibility between the obtained results and the natural behavior of the system shows that the suggested model can unveil a high-degree of complexity of the system’s internal dynamics under thermoelastic loading, which would otherwise be overlooked by other conventional models. In addition, the obtained separate energetic outlook of the suggested model offers a
considerable potential for the development of more control-oriented strategies that can address the
individual dynamics of each of the participating domains, instead of the total dynamics of the multi-
physical system.

REFERENCES