Ramsey numbers of trees versus fans

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For two given graphs $G_1$ and $G_2$, the Ramsey number $R(G_1, G_2)$ is the smallest integer $N$ such that, for any graph $G$ of order $N$, either $G$ contains $G_1$ as a subgraph or the complement of $G$ contains $G_2$ as a subgraph. Let $T_n$ be a tree of order $n$, $S_n$ a star of order $n$, and $F_m$ a fan of order $2m + 1$, i.e., $m$ triangles sharing exactly one vertex. In this paper, we prove that $R(T_n, F_m) = 2n - 1$ for $n \geq 3m^2 - 2m - 1$, and if $T_n = S_n$, then the range can be replaced by $n \geq \max\{m(m - 1) + 1, 6(m - 1)\}$, which is tight in some sense.

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1. Introduction

In this paper we deal with finite simple graphs only. For a nonempty proper subset $S \subseteq V(G)$, let $G[S]$ and $G - S$ denote the subgraph induced by $S$ and $V(G) - S$, respectively. Let $N_G(v)$ be the set of all the neighbors of a vertex $v$ that are contained in $S$, $N_G[v] = N_G(v) \cup \{v\}$ and $d_G(v) = |N_G(v)|$. If $S = V(G)$, we write $N(v) = N_G(v)$, $N[v] = N(v) \cup \{v\}$ and $d(v) = d_G(v)$. For two vertex-disjoint graphs $G_1$ and $G_2$, $G_1 \cup G_2$ denotes their disjoint union and $G_1 + G_2$ is the graph obtained from $G_1 \cup G_2$ by joining every vertex of $G_1$ to every vertex of $G_2$. We use $mG$ to denote the union of $m$ vertex-disjoint copies of $G$. A path, a star, a tree, a cycle and a complete graph of order $n$ are denoted by $P_n$, $S_n = K_1 + (n - 1)K_1$, $T_n$, $C_n$ and $K_n$, respectively. A book $B_n = K_1 + nK_1$, i.e., it consists of $n$ triangles sharing exactly one common edge, and a fan $F_n = K_1 + nK_2$, i.e., it consists of $n$ triangles sharing exactly one common vertex. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum and minimum degree of a graph $G$.

Given two graphs $G_1$ and $G_2$, the Ramsey number $R(G_1, G_2)$ is the smallest integer $N$ such that, for any graph $G$ of order $N$, either $G$ contains $G_1$ as a subgraph or $\overline{G}$ contains $G_2$ as a subgraph, where $\overline{G}$ is the complement of $G$. If both $G_1$ and $G_2$ are complete graphs, then $R(G_1, G_2)$ is called a classical Ramsey number, otherwise it is called a generalized Ramsey number. Because of the extreme difficulty encountered in the determination of classical Ramsey numbers, Chvátal and Harary [10–12] in a series of papers suggested studying generalized Ramsey numbers, both for their own sake, and for the light they might shed on classical Ramsey numbers. The following is a celebrated early result on generalized Ramsey numbers due to Chvátal.

\textbf{Theorem 1 (Chvátal [9])}. $R(T_n, K_m) = (n - 1)(m - 1) + 1$ for all positive integers $m$ and $n$.

Let $H$ be a connected graph of order $p$, $\chi(G)$ the chromatic number of $G$ and $s(G)$ the chromatic surplus of $G$, i.e., the minimum number of vertices in some color class under all proper vertex colorings with $\chi(G)$ colors. Based on Chvátal’s result, Burr [4]...
established the following general lower bound for $R(H, G)$ when $p \geq s(G)$: $R(H, G) \geq (p - 1)(\chi(G) - 1) + s(G)$. He also defined $H$ to be $G$-good in case equality holds in this inequality. By Theorem 1, it is easy to see that $T_n$ is $K_n$-good. This raises the natural questions whether and when $T_n$ is $G$-good if $G$ consists of $\ell$ complete graphs $K_m$ sharing exactly one vertex. A special case of the question is whether $T_n$ is $F_{\ell}$-good. Another nature question is for what graphs $G$, $T_n$ is $G$-good.

In 1982, Burr et al. determined the Ramsey numbers of sufficiently large trees versus odd cycles, by showing that $T_n$ is $C_m$-good for odd $m \geq 3$ and $n \geq 756m^{10}$.

**Theorem 2** (Burr et al. [5]). $R(T_n, C_m) = 2n - 1$ for odd $m \geq 3$ and $n \geq 756m^{10}$.

In 1988, Erdős et al. confirmed the Ramsey numbers of relatively large trees versus books, by showing that $T_n$ is $B_m$-good for $n \geq 3m - 3$, a result that we will use in our proof of Lemma 2 in the next section.

**Theorem 3** (Erdős et al. [13]). $R(T_n, B_n) = 2n - 1$ for $n \geq 3m - 3$.

Other results on Ramsey numbers concerning trees can be found in [1–3, 6–8, 14], see [15] for a survey. In this paper, we first show that $S_n$ is $F_{\ell}$-good for all integers $n \geq \max(m(m - 1) + 1, 6(m - 1))$, by proving the following result.

**Theorem 4.** $R(S_n, F_m) = 2n - 1$ for $n \geq m(m - 1) + 1$ and $m \neq 3, 4, 5$, and the lower bound $n \geq m(m - 1) + 1$ is best possible. $R(S_n, F_m) = 2n - 1$ for $n \geq 6(m - 1)$ and $m = 3, 4, 5$.

We postpone the proof of Theorem 4 to the last section. Next we show that $T_n$ is $F_{\ell}$-good for positive integers $n \geq 3m^2 - 2m - 1$, which is the main theorem of our paper.

**Theorem 5.** $R(T_n, F_m) = 2n - 1$ for all integers $n \geq 3m^2 - 2m - 1$.

We also postpone the proof of Theorem 5 to the last section. We next show that the following more general result can be obtained from Theorem 5 by induction.

**Corollary 1.** $R(T_n, K_{\ell - 1} + mK_2) = \ell(n - 1) + 1$ for $\ell \geq 2$ and $n \geq 3m^2 - 2m - 1$.

**Proof.** By Theorem 5, the statement is valid for $\ell = 2$. Assume that $k \geq 3$ and that the statement holds for all integers $\ell$ with $2 \leq \ell < k$. We prove that it also holds for $\ell = k$.

Since $KK_{n-1}$ contains no $T_n$ and its complement contains no $K_{k-1}$, hence no $K_{k-1} + mK_2$, we have $R(T_n, K_{k-1} + mK_2) \geq k(n-1) + 1$. Let $G$ be a graph of order $k(n-1) + 1$. If $\delta(G) \geq n-1$, then by the following folklore lemma that is straightforward to prove using a Greedy approach, $G$ contains $T_n$ and the proof is complete. We present the lemma in a more specific form since we will use it in this form in the sequel.

**Lemma 1.** Let $G$ be a graph with $\delta(G) \geq k$, and let $u \in V(G)$. Let $T$ be a tree of order $k + 1$ with $v \in V(T)$. Then $T$ can be embedded into $G$ in such a way that $v$ is mapped to $u$.

Let us now assume that $\delta(G) \leq n - 2$. Then $\Delta(G) \geq (k-1)(n-1) + 1$. Let $v$ be a vertex with $d_G(v) = \Delta(G)$. Then, by the induction hypothesis either $G[N_G(v)]$ contains a $T_n$, or $G[N_G(v)]$ contains a $K_{k-2} + mK_2$, which together with $v$ forms a $K_{k-1} + mK_2$. This completes the proof of Corollary 1.

We finish this section by posing a conjecture on the best possible lower bound for $n$ for which $T_n$ is $F_{\ell}$-good.

**Conjecture 1.** $R(T_n, F_m) = 2n - 1$ for $n \geq m^2 - 1$.

Let $G$ be any given graph. It is believed that $R(T_n, G) \leq R(S_n, G)$ in general, and all known results point in this direction. Based on this and Theorem 4, we believe that the above conjecture holds, at least for $m \geq 6$.

2. Two preliminary lemmas

In the next section we use the following lemma in our proof of Theorem 4. It is the special case of the statement of Theorem 4 when $m = 2$.

**Lemma 2.** $R(S_n, F_2) = 2n - 1$ for $n \geq 3$.

**Proof.** The lower bound $R(S_n, F_2) \geq 2n - 1$ is implied by the fact that $2K_{n-1}$ contains no $S_n$ and its complement contains no triangle, hence no $F_2$. It remains to prove that $R(S_n, F_2) \leq 2n - 1$ for $n \geq 3$.

Let $G$ be a graph of order $2n - 1$. Suppose that $G$ contains no $F_2$ and $\overline{G}$ has no $S_n$. Then $\Delta(\overline{G}) \leq n - 2$ and so $\delta(G) \geq n$. By Theorem 3, $G$ contains $B_2$. Let $x_1x_2x_3x_4$ be a $C_4$ with diagonal $x_2x_3$ in $G$. Set $X = \{x_1, x_2, x_3, x_4\}$ and $Y = V(G) - X$. If $n = 3$, then $|Y| = 1$ and the vertex in $Y$ has at least three neighbors in $X$, and so $G$ has $F_2$, a contradiction. Hence, $n \geq 4$. If $x_1x_3 \not\in E(G)$, then $N_Y(x_1) \cap N_Y(x_3) = \emptyset$ for $1 \leq i < j \leq 4$, otherwise $G$ contains $F_2$. Thus, we have $4(n - 2) \leq \sum_{k=1}^4 d_Y(x_k) + 4 \leq 2n - 1$, which implies that $n \leq 3$, a contradiction. If $x_1x_3 \not\in E(G)$, then since $G$ has no $F_2$, we get that $N_Y(x_1) \cap N_Y(x_i) = \emptyset$ for $i = 2, 4$ and $N_Y(x_1)$ is an independent set of cardinality at least $n - 2$. In this case, we have $d(y) \leq n - 1$ for any $y \in N_Y(x_1)$, which contradicts that $\delta(G) \geq n$. ■
We use the following lemma in our proof of Theorem 5. It deals with Ramsey numbers of trees versus $mk_2$ instead of $F_m$ and might be of some interest by itself.

**Lemma 3.** $R(T_n, mk_2) = n + m - 1$ for $n \geq 4(m - 1)$.

**Proof.** The result is trivial for $m = 1$, thus we assume that $m \geq 2$. Since $K_{n-1} \cup (m-1)K_1$ contains no $T_n$ and its complement contains no $mk_2$, we conclude that $R(T_n, mk_2) \geq n + m - 1$. It remains to prove that $R(T_n, mk_2) \leq n + m - 1$ for $n \geq 4(m - 1)$.

Let $G$ be a graph of order $n + m - 1$, and suppose to the contrary that neither $G$ contains a $T_n$ nor $G$ contains $mk_2$. Let $M = \{x_1y_1, \ldots, x_ky_k\} \subseteq E(G)$ be a maximum matching in $G$ and $X = V(G) - V(M)$. Then, obviously $t \leq m - 1$ since $G$ contains no $mk_2$, and $|X|$ is a complete graph by the maximality of $M$. Assume without loss of generality that $d_x(x_i) \leq d_x(y_i)$ for $1 \leq i \leq t$ in $G$. By the maximality of $M$, $d_x(x_i) \leq 1$ for $1 \leq i \leq t$ in $G$. Let $Y$ be the subset of $X$ containing all adjacent vertices of $\{x_1, \ldots, x_t\}$ in $G$. Then, by the previous arguments $|Y| \leq t \leq m - 1$. Since $T_n$ is a bipartite graph, we may assume without loss of generality that $V(T_n) = (X', Y')$ with $|X'| \geq |Y'|$. Since $n \geq 4(m - 1)$, we get that $|Y'| \geq n/2 \geq 2(m - 1) \geq |Y| + t$. Now we can embed $T_n$ into $G$ using the following procedure. First map $|Y'| + t$ vertices of $Y'$ to $Y \cup \{x_1, \ldots, x_t\}$ arbitrarily, and then map the remaining vertices of $T_n$ to $X - Y$ arbitrarily. This is possible because $|X| + t = n + m - 1 - 2t + t = n + m - (t + 1) \geq n$ and every vertex of $X - Y$ is adjacent to every vertex of $X \cup \{x_1, \ldots, x_t\}$ except itself. Thus, $G$ contains $T_n$, a contradiction. This completes the proof of Lemma 3. ■

3. Proofs of the main results

We use the lemmas of the previous sections to prove our main results in separate subsections.

3.1. Proof of Theorem 4

The result is easy to prove for $m = 1$ and in this case follows also from Theorem 1, and it holds for $m = 2$ by Lemma 2, thus we may assume that $m \geq 3$.

We first go on to show that if $n \leq m(m - 1)$, then $R(S_n, F_m) \geq 2n$, showing that the lower bound $n \geq m(m - 1) + 1$ is in some sense best possible. Since $K_{m-1}$ contains no $F_m$ and its complement contains no $S_n$, we have $R(S_n, F_m) \geq 2m$, so we only need to consider the case that $n \geq m + 1$. There exist positive integers $p, q$ such that $n = pm + q$ and $1 \leq q \leq m$.

Let $H = pS_m \cup S_1$ if $q \neq 1$, and $H = (p - 1)S_m \cup S_m - 1 \cup S_2$ if $q = 1$. Since $n \leq m(m - 1)$, then $p \leq m - 2$. It is easy to check that $H$ is a graph of order $n$ with $\delta(H) \geq 1$, and that $H$ contains neither $S_{m+1}$ nor $mk_2$. Let $H' = K_n - 1 \cup \overline{T}$. Then $H'$ contains no $S_n$ and $\overline{H}$ contains $F_m$. Thus, if $n \leq m(m - 1)$, then $R(S_n, F_m) \geq 2n$.

It remains to show that $R(S_n, F_m) = 2n - 1$ for $n \geq \max\{m(m - 1) + 1, 6(m - 1)\}$ and $m \geq 3$. First we note that since $2K_{m-1}$ contains no $S_n$ and its complement contains no $F_m$, we conclude that $R(S_n, F_m) \geq 2n - 1$.

To prove $R(S_n, F_m) \leq 2n - 1$, let $G$ be a graph of order $2n - 1$ and suppose to the contrary that $G$ contains no $F_m$ and $\overline{G}$ contains no $S_n$. Then $\Delta(\overline{G}) \leq n - 2$ and $\delta(G) \geq n$. For any vertex $u$ of $V(G)$, let $M_u \subseteq E(\overline{G})$ be a maximum matching in $G[N(u)]$ and $X_u = N(u) - V(M_u)$. Then, obviously $G[N[u]]$ contains no edges, and $|M_u| \leq m - 1$; otherwise $G[N[u]]$ contains an $F_m$, a contradiction. Moreover, by the maximality of $M_u$, for $xy \in M_u$, if $d_x(x) > 2$, then $d_x(y) = 0$; and if $d_x(x) = d_x(y) = 1$, then $x$ and $y$ are adjacent to the same vertex in $X_u$. Let $Y_u \subseteq V(M_u)$ be the set of vertices that have at least two neighbors in $X_u$, and let $Z_u = N(u) - Y_u$. It is obvious that $|Y_u| \leq m - 1$ and $|Z_u| \geq n - m - 1$.

Since $X_u \subseteq Z_u$ and $|Z_u| \geq n - 2(m - 1) \geq m$, there exists a vertex $v \in X_u$ with $d_{X_u}(v) = 0$. We define $M_v, X_v, Y_v, Z_v$ in a completely analogous way. Since $d_{Z_u}(v) = 0$ and $Z_v \subseteq N(v)$, we get that $Z_u \cap Z_v = \emptyset$. Hence, $X_u \cap X_v = \emptyset$. We first prove the following two claims.

**Claim 1.** Let $V_1 = \{w \mid |X_u \cap X_v| \geq |X_u| - 2m + 3$$\}$ and $V_2 = \{w \mid |X_u \cap X_v| \geq |X_u| - 2m + 3$$\}$ and $X_u \cap X_v = \emptyset$. Then for any vertex $w$ of $V(G)$, either $w \in V_1$ or $w \in V_2$. Moreover, $Z_u \subseteq V_1, Z_v \subseteq V_2$.

**Proof.** For any vertex $w$ of $V(G)$, if $w \cap X_u = \emptyset$ and $X_u \cap X_v = \emptyset$, then $2n - 1 \geq |X_u| + |X_v| + |X_u| \geq (n - 2(m - 1) - 1$, and hence $n \leq 6(m - 1) - 1$, a contradiction. Thus, either $X_u \cap X_v \neq \emptyset$ or $X_u \cap X_v \neq \emptyset$. If $w \not\in X_u \cap X_v$, then both $G[X_u]$ and $G[X_v]$ are edgeless graphs, then for any vertex $x \in X_u \cap X_v$, we have $d(z) \geq |X_u| + |X_v| - |X_u| \cap X_v = 1$ in $\overline{\overline{G}}$. Since $\min(\Delta(\overline{G})) \leq n - 2$, we obtain $|X_u| \cap X_v \geq |X_u| + |X_v| - 1 - (n - 2)$. Hence, $|X_u| \cap X_v \geq |X_u| - 2m + 3$ and $|X_v| \cap X_v \geq |X_v| - 2m + 3$. For the same reason, if $w \not\in X_u \cap X_v$, then $|X_u| \cap X_v \geq |X_u| - 2m + 3$ and $|X_v| \cap X_v \geq |X_v| - 2m + 3$. If both $X_u \cap X_v \neq \emptyset$ and $X_u \cap X_v \neq \emptyset$, then $|X_u| \cap X_v \geq |X_u| \cap X_v \geq 2(|X_u| - 2m + 3)$, and hence $|X_u| \leq 4m - 6$, which contradicts $|X_u| \geq n - 2(m - 1) \geq 4m - 4$. Therefore, for any vertex $w$ of $V(G)$, either $w \in V_1$ or $w \in V_2$.

Any vertex $w$ of $Z_u$ has at most one adjacent vertex in $X_v$, hence $w \in V_1$. Thus, $Z_v \subseteq V_1$. By symmetry, $Z_u \subseteq V_2$. ■

**Claim 2.** For any two vertices $w_1, w_2 \in V_1$, $|X_{w_1} \cap X_{w_2}| \geq 2m - 1$. For any two vertices $w_3, w_4 \in V_2$, $|X_{w_3} \cap X_{w_4}| \geq 2m - 1$.

**Proof.** It is sufficient to prove the first statement. For any two vertices $w_i, w_j \in V_1$, since $|X_{w_i} \cap X_{w_j}| \geq |X_{w_i}| - 2m + 3$ for $i = 1, 2$, we get that $|X_{w_1} \cap X_{w_2}| \geq |X_{w_1} \cap X_{w_2}| \geq |X_{w_1} \cap X_{w_2}| \geq 2m - 1$. Since both $G[X_{w_1}]$ and $G[X_{w_2}]$ are edgeless graphs, for any vertex $x \in X_{w_1} \cap X_{w_2}$, we have $d(z) \geq |X_{w_1}| + |X_{w_2}| - |X_{w_1} \cap X_{w_2}| = 1$ in $\overline{\overline{G}}$. Since $\min(\Delta(\overline{G})) \leq n - 2$, we obtain that $|X_{w_1} \cap X_{w_2}| \geq |X_{w_1}| + |X_{w_2}| = 2m - 1$. ■
By Claim 1, every vertex belongs to either $V_1$ or $V_2$, but not both. Since $|V(G)| = 2n - 1$, we have $|V_1| \geq n$ or $|V_2| \geq n$. Without loss of generality, we may assume that $|V_1| \geq n$. For any vertex $z$ of $V_1$, if $d_{V_1}(z) \geq m$, we choose $m$ adjacent vertices of $z$ from $V_1$, denoted by $z_1, \ldots, z_m$. By Claim 2, for $1 \leq i \leq m$, $z_i$ has at least $m$ adjacent vertices in $X_s - \{z_1, \ldots, z_m\}$. Thus, we may find a matching of $m$ edges in $G(N(z))$, which together with $z$ forms an $F_m$, a contradiction. Therefore, for any vertex $z$ of $V_1$, we have $d_{V_1}(z) \leq m - 1$. If $|Z_s| \geq n$, since $X_s \subseteq Z_s$ and $|X_s| \geq n - 2(m - 1) \geq m$, there exists a vertex of degree 0 in $G[Z_s]$, that is, $G[Z_s]$ contains a vertex of degree at least $n - 1$, a contradiction. This implies that $|Z_s| \leq n - 1$. Since $Z_s \subseteq V_1$, we choose a subset of $V_1$ containing $Z_s$ and any $n - |Z_s|$ vertices of $V_1 - Z_s$. For simplicity, this subset of $V_1$ is also denoted by $V_1$ in the sequel. Thus, $|V_1| = n$.

In the remainder, we prove that there exists a vertex $z_0$ of $V_1$ such that $d_{V_1}(z_0) = 0$ in $G$, and then $d_{V_1}(z_0) = n - 1$ in $\overline{G}$, which is a contradiction. Since $|Z_s| \geq n - m + 1$, we distinguish three cases: $|Z_s| = n - m + 3$ and $|Z_s| = n - m + 2$, separately.

If $|Z_s| = n - m + 3$, $X_s$ contains at most $m - 1$ vertices which are adjacent to $Z_s - X_s$, and every vertex of $V_1 - Z_s$ is adjacent to at most $m - 1$ vertices of $X_s$. Since $|X_s| \geq n - 2(m - 1)$, $|V_1 - Z_s| \leq m - 3$ and $n - 2(m - 1) - (m - 1) - (m - 3)(m - 1) \geq 1$, so we may find the required $z_0$ in $X_s$, that is, with $d_{V_1}(z_0) = 0$ in $G$.

If $|Z_s| = n - m + 1$, then $|Y_s| \geq n - |Z_s| = m - 1$, and by the maximality of $M_s$, $G[Z_s]$ is an edgeless graph. Since every vertex of $V_1 - Z_s$ is adjacent to at most $m - 1$ vertices of $Z_s$, $|V_1 - Z_s| = m - 1$ and $n - m + 1 - (m - 1)^2 \geq 1$, so we may find the required $z_0$ in $Z_s$, that is, with $d_{V_1}(z_0) = 0$ in $G$.

We recall that we have shown that $R(S_n, F_m) \geq 2n$ for $n \leq m(m - 1) + 1$ is best possible for $m \geq 6$.

3.2. Proof of Theorem 5

Recall that we want to prove that $R(T_n, F_m) = 2n - 1$ for all integers $n \geq 3m^2 - 2m - 1$. The lower bound $R(T_n, F_m) \geq 2n - 1$ is implied by the fact that $2K_{n-1}$ contains no $T_n$ while its complement contains no $F_m$. Now we prove the upper bound.

We may assume that $m \geq 2$ since the result is easy to prove for $m = 1$ and in this case follows also from Theorem 1. Let $G$ be a graph of order $2n - 1$ with $n \geq 3m^2 - 2m - 1$ and $m \geq 2$. Suppose to the contrary that $G$ contains no $F_m$ and its complement contains no $T_n$. We first claim that $\Delta(G) \leq n + m - 2$. If not, let $u$ be a vertex with $\Delta(G) = \Delta(G) \leq n + m - 1$. Since $n \geq 3m^2 - 2m - 1$ and $m \geq 2$, this implies that $n \geq 4(m - 1)$. By Lemma 3, either $G(N(u))$ contains $mK_2$, which together with $u$ forms an $F_m$, or $|G[N(u)]| \leq n - m + 2$. Therefore, we have $\Delta(G) \leq n + m - 2$.

Next we prove that Theorem 5 holds when $T_n(\Delta(G)) \geq 13n/24$. Let $u$ be a vertex of largest degree in $T_n$, let $A$ denote the set of vertices of $T_n$ that are adjacent to $u$ and have degree one in $T_n$, and let $B$ denote the set of vertices of $T_n$ that are adjacent to $u$ and have degree at least two in $T_n$. Then, obviously since $T_n$ is a tree, $|V(T_n)| \geq 1 + |A| + 2|B|$ and $\Delta(T_n) = |A| + |B|$. Since $|V(T_n)| = n$ and we assume that $\Delta(T_n) \geq 13n/24$, we obtain that $|A| + n + 1 = 2|A| + 2|B| = 1 + 2\Delta(T_n) \geq 1 + 13n/12$, hence $|A| \geq n/12 + 1$. Then $T_n - A$ is a tree of order at most $11n/12 - 1$. We want to apply Lemma 1 to embed $T_n - A$ in $\overline{G}$ such that $u$ is mapped to the vertex of degree $n - 1$ of an $S_n$. Since $|V(T_n)| \leq 11n/12 - 1$, it is sufficient to show that $\delta(\overline{G}) \geq 11n/12 - 2$ and that $\overline{G}$ contains an $S_n$.

Since $\Delta(G) \leq n + m - 2$, we get that $\Delta(G) \geq (2n - 1) - 1 - (n + m - 2) = n - m$. Using $m \geq 2$, it is easy to check that $3m^2 - 2m - 1 \geq 12m - 24$. By the condition of the theorem, $n \geq 3m^2 - 2m - 1 \geq 12m - 24$, so $n/12 \geq m - 2$, and hence $n \geq 11n/12 - 2$. Furthermore, again using $m \geq 2$, $3m^2 - 2m - 1 \geq \max(m(m - 1) + 1, 6(m - 1))$. By Theorem 4, $\overline{G}$ contains an $S_n$. By Lemma 1, $T_n - A$ can be embedded in $\overline{G}$ such that $u$ is mapped to the vertex with degree $n - 1$ of the $S_n$. Because $u$ now has at least $n - 1$ adjacent vertices in $\overline{G}$, the embedding of $T_n - A$ can easily be extended to $T_n$ in $\overline{G}$. This contradicts the assumption that $G$ contains no $T_n$, completing this case. So, in the remainder of the proof we assume that $\Delta(T_n) < 13n/24$.

By Lemma 1, $\delta(\overline{G}) \leq |V(T_n)| - 2 = n - 2$; otherwise we can embed $T_n$ in $\overline{G}$. So we obtain that $\Delta(G) \leq n$. Let $x$ be a vertex with $\Delta(G) = \Delta(G) \geq n$, let $M = \{x_1y_1, \ldots, x_1y_2\} \leq E(G[N(x)])$ be a maximum matching in $G[N(x)]$, and let $U = V(G[N(x)]) \setminus V(M)$. Then $G[U]$ is an edgeless graph, and $t \leq m - 1$; otherwise $G[N(x)]$ contains $mK_2$, which together with $x$ forms an $F_m$, a contradiction. Without loss of generality, suppose that $d_{U}(y_1) \leq d_{U}(y_i)$ for $1 \leq i \leq t$, and suppose that $k$ and the order of vertices are chosen such that $d_{U}(y_1) \leq 1$ for $1 \leq i \leq k$, and $d_{U}(y_i) \geq 2$ for $k + 1 \leq i \leq t$. (We assume that the degenerate cases that all $d_{U}(y_i) \leq 1$ or all $d_{U}(y_i) \geq 2$ do not occur, but these can be dealt with similarly.) By the maximality of $M$, $d_{U}(x_1) = 0$ for $k + 1 \leq i \leq t$, $d_{U}(x_1) \leq 1$ for $1 \leq i \leq k$, and if $d_{U}(x_1) \neq 0$, then $x_i$ and $y_i$ are adjacent to the same vertex of $U$. Let $Y$ consist of the set $V(M) \setminus \{y_{k+1}, \ldots, y_t\}$ and its adjacent vertex set in $U$, and let
Therefore, there exists a vertex \( \Delta \) components is at most \( k \). If not, each component of \( Y \) of order at least two, then the number of non-trivial components is at most \( k \). Now we can embed \( T' \) in \( \overline{G}(X \cup Y) \) through the following procedure. First map \( w_1 \) to \( w_2 \); then map \(|Y| \) vertices of \( Y_1 \) to \( Y \) arbitrarily. Finally, map the remaining vertices of \( T' \) to \( X \) arbitrarily. Because in \( \overline{G} \) every vertex of \( X \) is adjacent to every vertex of \( X \cup Y \) except itself, the embedding can succeed. 

If \(|X| + |Y| \geq n - 1\), then by Claim 3, \( \overline{G} \) contains \( T_n \), a contradiction. So we may assume \(|X| + |Y| \leq n - 2\). Let \( T' \) be a largest subtree of \( T_n \) that can be embedded in \( G \). Then \( T' \) is a proper subgraph of \( T_n \). This implies there exists a vertex in \( T' \), say \( x' \), such that \( x' \) is adjacent to every vertex of \( V(G) - V(T') \) in \( G \). Hence, \( d_{G-V(T')} (x') \geq n \).

In \( G[N(x')] - (X \cup Y) \), we define \( M', U', t', k', X', Y' \) in a completely analogous way as we have defined \( M, U, t, k, X, Y \) in \( G[N(x)] \). Now we distinguish two cases.

Case 1. In \( \overline{G}, d_x(z) \geq m/2 \) for some \( z \in X' \), or \( d_Y(z) \geq m/2 \) for some \( z \in X \).

By symmetry, we may assume that \( d_x(z) \geq m/2 \) for some \( z \in X' \) in \( \overline{G} \). For \( v \in V(T_n) \), let \( H_1, \ldots, H_p \) be all the components of \( T_n - v \) with at most \( m - 1 \) vertices, and ordered in such a way that \(|v| \geq |V(H_1)| \geq \cdots \geq |V(H_p)| \). We distinguish two subcases.

Subcase 1.1. There exists a vertex \( v \) of \( T_n \) such that \( \sum_{i=1}^p |V(H_i)| \geq m - 1 \), where \( p = \min\{m/2, \ell\} \).

We give an embedding of \( T_n \) in \( \overline{G} \). First we map \( v \) to \( z \). Let \( v_0 \) be the vertex of \( H_1 \) adjacent to \( v \) in \( T_n \). Since \( d_x(z) \geq m/2 \) and \( p \leq \lfloor m/2 \rfloor \), we map \( v_1, \ldots, v_p \) sequentially to the adjacent vertices of \( z \) in \( \overline{G}[X] \). Since \( |X| \geq n - 2 \), we have \(|X| \geq \lfloor m/2 \rfloor (m - 1) \geq \sum_{i=1}^p |V(H_i)| \). Since \( \overline{G}[X] \) is a complete graph, \( H_1, \ldots, H_p \) can be embedded in \( \overline{G}[X] \) easily. Since \( \sum_{i=1}^p |V(H_i)| \geq m - 1 \), \( T_n - \bigcup_{i=1}^p V(H_i) \) is a tree of order at most \( n - m + 1 \). Since \(|X| + |Y| \geq n - m + 1 \), we have \(|X'| + |Y'| \geq n - m + 1 \) by symmetry. By Claim 3 and the symmetry of \( \overline{G}[X \cup Y] \) and \( \overline{G}[X' \cup Y'] \), \( \overline{G}[X' \cup Y'] \) contains \( T_n - \bigcup_{i=1}^p V(H_i) \) such that \( v \) is mapped to \( z \). Therefore, \( \overline{G} \) contains \( T_n \), a contradiction.

Subcase 1.2. For any vertex \( v \) of \( T_n \), \( \sum_{i=1}^p |V(H_i)| < m - 1 \), where \( p = \min\{m/2, \ell\} \).

We first show that we may assume that for any vertex \( v \in V(T_n) \), the largest component of \( T_n - v \) is of order at least \( m \). If not, each component of \( T_n - v \) is of order at most \( m - 1 \). Since Subcase 1.1 does not occur and each nontrivial component is of order at least two, then the number of nontrivial components is at most \( m/2 - 1 \), and the total order of the nontrivial components is at most \( m - 2 \). Thus, the total order of the trivial components is at least \( n - m + 1 \). This implies that \( d(v) \geq n - m - 1 \). Using \( n \geq 3m^2 - 2m - 1 \) and \( m \geq 2 \), we easily obtain that \( d(v) \geq 13n/24 \), but we have already shown that Theorem 5 holds when \( \Delta(T_n) \geq 13n/24 \). Thus, henceforth we may assume that for any vertex \( v \in V(T_n) \), the largest component of \( T_n - v \) is of order at least \( m \).

Choose a vertex \( v \) from \( T_n \) such that the order of the largest component of \( T_n - v \) is as small as possible. Let \( H_0 \) be a largest component of \( T_n - v \) with \( v_0 \in V(H_0) \) being adjacent to \( v \) in \( T_n \). Then we claim that \(|V(H_0)| \leq n - m - 1 \). Suppose to the contrary that \(|V(H_0)| \geq n - m + 1 \). By the choice of \( v \), the largest component of \( T_n - v_0 \) has at least \( m/2 + 2 \) vertices, so this is the component of \( T_n - v_0 \) containing \( v \). In that case, \(|V(H_0)| \leq m - 2 \), a contradiction to our assumption. Therefore, there exists a vertex \( v \) such that \(|v| \leq d_{\overline{G}}[X|Y]| = n - m - 1 \), where \( H_0 \) is the largest component of \( T_n - v \).

Let \( zz' \in E(\overline{G}) \) with \( z \in X' \) and \( z' \in X \). By symmetry and by Claim 3, we may embed \( H_0 \) in \( \overline{G}[X \cup Y] \) such that \( v_0 \) is mapped to \( z' \), and \( T_n - V(H_0) \) in \( \overline{G}[X' \cup Y'] \) such that \( v \) is mapped to \( z \). Thus, \( \overline{G} \) contains \( T_n \), a contradiction. This completes Case 1.

Case 2. In \( \overline{G}, d_x(z) < m/2 \) for every \( z \in X' \), and \( d_Y(z) < m/2 \) for every \( z \in X \).

First consider an arbitrary vertex \( v \in V(G) - (X \cup Y \cup X' \cup Y') \). Suppose \( d_x(v) \geq \lfloor 3m/2 \rfloor - 1 \) and \( d_Y(v) \geq \lfloor 3m/2 \rfloor - 1 \). Then, since every vertex of \( N_X(v) \) has at most \( \lfloor m/2 \rfloor - 1 \) adjacent vertices of \( X \) in \( \overline{G} \), every vertex of \( N_Y(v) \) has at most \( m/2 \) adjacent vertices of \( N_X(v) \). Thus, in that case we may find a matching of \( m \) edges in \( N_{X \cup \overline{X}}(v) \), which together with \( v \) forms an \( f_m \), a contradiction. Therefore, for every vertex \( v \in V(G) - (X \cup Y \cup X' \cup Y') \), if \( d_x(v) \leq \lfloor 3m/2 \rfloor - 2 \), then put \( v \) in \( Z' \); if this is not the case, then put \( v \) in \( Z' \). Now \((X, Y, Z, X', Y', Z')\) is a partition of \( \overline{G} \). Since \(|V(G)| = 2n - 1\), either \(|X| + |Y| + |Z| \geq n\), or \(|X'| + |Y'| + |Z'| \geq n\). Without loss of generality, assume that \(|X| + |Y| + |Z| \geq n\). Let \( Z'\) be a subset of \( Z \) with exactly \( n - |X| - |Y| \leq t \) vertices. Then every vertex of \( Z'\) has at most \( \lfloor 3m/2 \rfloor - 2 \) adjacent vertices in \( X \). Since \( n \geq 3m^2 - 2m - 1, |X| - (\lfloor 3m/2 \rfloor - 2) |Z'| \geq (n - 2t - k) - (\lfloor 3m/2 \rfloor - 2)(t - k) \geq n - \lfloor 3m/2 \rfloor (t - n \lfloor 3m/2 \rfloor (m - 1) \geq n/2 \).

Since \( T_n \) is a bipartite graph, we may assume \( V(T_n) = (X_2, Y_2) \) and \( X_2 \subseteq Y_2 \). Now we can embed \( T_n \) in \( \overline{G}[X \cup Y \cup Z'] \) through the following procedure. First map \( |Y| + |Z'| + |N_X(Z')| \) vertices of \( Y_2 \) to \( Y \cup Z' \cup N_X(Z') \) arbitrarily; then map the remaining vertices of \( T_n \) to \( X - N_X(Z') \) arbitrarily. Because in \( \overline{G} \), every vertex of \( X - N_X(Z') \) is adjacent to every vertex of \( X \cup Y \cup Z' \) except itself, and \(|X| - N_X(Z') \geq n/2 \), the embedding can succeed. Thus, \( \overline{G} \) contains \( T_n \), our final contradiction.
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