Embedded wave generation for dispersive surface wave models

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\textbf{A R T I C L E I N F O}

\begin{abstract}
This paper generalizes previous research on embedded wave generation in Boussinesq-type of equations for multi-directional surface water waves; the generation takes place by adding a suitable source term to the equations. Accurate generation is important to prevent influx errors in simulated waves downstream. For numerical implementations it may be a useful alternative to boundary influx methods since it is relatively easy to implement and will account accurately for the dispersive properties of the model. The source functions are unique only when the spatial and temporal constituents satisfy the dispersion relation of the model; this ambiguity can be used to choose the spatial extent over which the generation is applied by adjusting the given input signal. Elevation and velocity type of generation can then produce waves running forward or partly forward and partly backward as desired. The sources, derived for linear models, can also generate high waves in nonlinear equations provided an adjustment zone in which the nonlinearity grows gradually is used. Results of simulations are shown for various cases, including a focusing wave and oblique wave interaction.
\end{abstract}

1. Introduction

Wave models of Boussinesq type for the evolution of surface waves on a layer of fluid describe the evolution with quantities at the free surface. These models have dispersive properties that are directly related to the – unavoidable – approximation of the interior fluid motion. Numerical implementations will have somewhat different dispersion, depending on the specific method of discretization. The initial value problem for such models does not cause much problems, since the description of the state variables in the spatial domain at an initial instant is independent of the specifics of the evolution model.

Quite different is the situation when waves have to be excited in a timely manner from points or lines. Such problems arise naturally when modelling uni- or multi-directional waves in a hydrodynamic laboratory or waves from the deep ocean to a coastal area. In these cases the waves can be generated by influx-boundary conditions, or by some embedded, internal, forcing. In all cases the dispersive properties (of the implementation) of the model are present in the details of the generation. Accurate generation is essential for good simulations, since slight errors will lead after propagation over large distances to large errors. For various Boussinesq type equations, internal wave generation has been discussed in several papers.

Improving the approach of Engquist and Majda (1977), who described the way how to influx waves at the boundary with the phase speed, Wei et al. (1999) considered the problem to generate waves from the y-axis under an angle $\theta$ with respect to the positive x-axis. They derived in an analytical way a spatially distributed source function method for the Boussinesq model of Wei and Kirby (1995) that is based on a spatially distributed source, with an explicit relation between the desired surface wave and the source function. Chawla and Kirby (2000) showed forward propagating influx. Kim et al. (2007) showed that for various Boussinesq models, it is possible to generate oblique waves using only a delta source function. Madsen and Sørensen (1992) used and formulated a source function for mild slope equations. In these papers, the results were derived for the linearized equations.

Different from embedded wave generation, in the so-called relaxation method the generation and absorption of waves is achieved by defining a relaxation function that grows slowly from 0 to 1 to target solution that has to be known in the relaxation area. The method, combined with a stream-function method (Fenton, 1988) to determine the target solution, has been used by e.g. Madsen et al. (2003), Fuhrman and Madsen (2006), Fuhrman et al. (2006), and Jamois et al. (2006); for an application of the method in other free surface models see Jacobsen et al. (2012).

This paper deals with embedded wave generation for which the wave elevation (or velocity) is described together with for- or backward propagating information at a boundary. Source functions for any kind of waves to be generated are derived for any dispersive equation, including the general case of dispersive Boussinesq equations. Consequently, the results are applicable for the equations considered in the references mentioned above, such as Boussinesq
A dispersive wave equation is determined by its dispersion relation, specifying the relation between the wave number $k$ and the frequency $\omega$ so that harmonic modes $\exp(ikx - \omega t)$ are physical solutions.

For a second order in time equation, the relation can be written as

$$\omega^2 = D(k)$$

where $D$ is a non-negative, even function. In modelling and simulating waves, the dispersion relation expresses the translation of the interior fluid motion to quantities at the surface, which implies a dimension reduction of one. Equations which model the waves with quantities in horizontal directions only are called Boussinesq-type of equations. The interior fluid motion in the layer below the free surface is then usually only approximately modelled. For linear waves, in the approximation of infinitesimal small wave heights, the exact dispersion relation $D_{ex}$ is given by

$$D_{ex}(k) = gk \tanh(kh)$$

with $g$ and $h$ being the gravitational acceleration and depth of the fluid layer respectively. Since this non-rational function has to be approximated by rational approximations for numerical methods except spectral methods, numerous approximations have been invented in such a way that errors from this approximation and errors from nonlinear effects are balanced for the type of waves to be investigated.

The second order equation corresponding to the dispersion relation $\omega^2 = D(k)$ can be written as

$$\ddot{\eta} + D\eta = 0$$

where $D$ is a pseudo-differential operator when $D$ is not a polynomial, but in all cases it is uniquely defined by multiplication in Fourier space as

$$D\eta(t) = D(k)\hat{\eta}(k)$$

From the nonnegativity and evenness of $D$, the operator $D$ will be a symmetric, positive definite operator. Defining the positive root of $D$, as the function $\Omega$,

$$\Omega(k) = \sqrt{D(k)}$$

introduce the odd function

$$\Omega_1(k) = \text{sign}(k)\Omega(k)$$

Then the wave exp $i(kx - \Omega(k)t) = \exp i(kx - C(k)t)$ is for all values of $k$ to the right travelling with positive phase speed $C(k) = \Omega(k)/k$; similarly $\exp i(kx + \Omega(k)t)$ is to the left travelling with speed $-C(k)$.

By defining the corresponding skew symmetric operator

$$A_1 \doteq i\Omega_1(k)$$

the operator $D$ can be factorized as

$$D = A_1^*A_1 = -(A_1)^2$$

The second order in time equation is then factorized as

$$(\ddot{\eta} + D)\eta = (\partial_t - A_1)(\partial_t + A_1)\eta$$

The first order in time operators describe to the right and left travelling waves, which are precisely the solutions of the unidirectional equations

$$(\partial_t + A_1)\eta_r = 0, \quad (\partial_t - A_1)\eta_l = 0$$

for which the dispersion relations are $\omega = \Omega_1(k)$ and $\omega = -\Omega_1(k)$ respectively. For construction of the embedded sources of the bi-directional equation, this factorization will be used.

In the following we will need the property that the function $D$ is monotonically increasing for $k > 0$, so that $\Omega_1(k)$ has a unique
inverse for all real $k$ which we will denote by $K_1$:
\[ \omega = \Omega_1(k) \iff k = K_1(\omega). \]
For later reference, recall that the group velocity is the even function given by
\[ V_g(k) = \frac{d\Omega_1(k)}{dk} \]
The exact dispersion given above corresponds to a monotone concave function $\Omega_1$, so that the phase velocity decreases for shorter waves; this will also be a reasonable assumption for approximations that are not only meant to be valid for long waves, such as the shallow water equations. Note furthermore that the scaling property of the exact dispersion relation and group velocity with depth is given by
\[ \Omega_1(k, h) = \sqrt{g} M(k h) \]
\[ V_g(k, h) = c_0 m(k h) \quad \text{with} \quad c_0 = \sqrt{gh} \]
respectively, where $m$ is the derivative of $M$. For reliable wave models with approximate dispersion, the same scaling properties will be satisfied, at least in a restricted interval of wave numbers.

In models that are used for analytic or numerical investigations, the approximation of the exact dispersion relation will satisfy in the relevant intervals the same scaling properties. As one example we mention the Variational Boussinesq Model (VBM) described in Adyia and van Groesen (2012) and Klopman et al. (2010). In that model, the dependence of the fluid potential in the vertical direction $z$ is prescribed by an a priori chosen function $F(z)$. The dispersion relation then reads
\[ \Omega_{\text{VBM}}(k) = c_0 k \sqrt{1 - \frac{(k \beta)^2}{h (k^2 + \gamma)}} \]
where $\alpha$, $\beta$ and $\gamma$ are coefficients given by
\[ \alpha = \int_{-h}^{0} F(z)^2 \, dz; \quad \beta = \int_{-h}^{0} F(z) \, dz; \quad \gamma = \int_{-h}^{0} (\partial_z F(z))^2 \, dz. \]
A flexible choice for $F(z)$ is to take the following explicit function:
\[ F(z) = \frac{\cosh(\kappa (z + h))}{\cosh(\kappa h)} - 1 \]
where $\kappa$ is a suitable effective wave number.

Another approximation is the shallow water (long wave) model where the dispersion relation is given as $\Omega_{\text{SW}} = c_0 k$. In Fig. 1 we show the plot of the exact dispersion relation and the exact group velocity together with the approximations described above.

In the following also the spatial inverse Fourier transform of the group velocity will be used, defined with a scaling factor as $\gamma(x, h) = \frac{V_g(k, h)}{2\pi}$
The scaling property of the group velocity implies that $\gamma(x, h)$ scales with depth like $\gamma(x; h) = \gamma(x/h; 1) / \sqrt{h}$. For later interest is especially that for increasing depth, the spatial extent of the area grows proportionally with $h$; see Fig. 2.

### 2.2. Influxing in uni-directional equations

Consider the first order in time uni-directional equation for to the right (positive $x$-axis) traveling waves
\[ \partial_t \eta = -A_1 \eta \]
The signaling problem for this equation is to find the solution $\zeta$ such that at one position, taken without restriction of generality to be $x = 0$, the surface elevation is prescribed by the given signal $s(t)$
\[ \begin{cases} \partial_t \zeta = -A_1 \zeta \\ \zeta(0, t) = s(t) \end{cases} \]
here and in the following it is assumed that the initial surface elevation and the signal vanish for negative time: $\zeta(x, 0) = 0$ and $s(t) = 0$ for $t \leq 0$. The solution of the signaling problem can be written explicitly as
\[ \zeta(x, t) = \Theta(x) \int s(\omega) e^{i K_1(x - \omega t) \omega} d\omega \]
with $\Theta(x)$ being the Heaviside function. Rewriting leads to the expression in which $s(t)$ appears explicitly
\[ \zeta(x, t) = \frac{1}{2\pi} \Theta(x) \int s(\tau) e^{i K_1(x - \tau - x t) \omega} d\omega \]
In this paper the solution of the signaling problem will be obtained by describing an influx in an embedded way. That is, for a forced problem of the form
\[ \begin{cases} \partial_t \eta = -A_1 \eta + S_1(x, t) \\ \eta(x, 0) = 0 \end{cases} \]
\[ (5) \]

Fig. 1. Plot of the dispersion relation (left panel) and the group velocity (right panel) as a function of wave number for depth 1 m. The solid curve is the exact dispersion and group velocity; the dash-dotted curve is the approximation for shallow water; the cross-dotted curve is the VBM-dispersion [1] with coefficient corresponding to $\kappa = 0.52$. 


the embedded source \( s(x,t) \) will be determined in such a way that the source contributes to the elevation at \( x=0 \) by an amount determined by the prescribed signal \( s(t) \).

For this first order uni-directional equation, a unique solution will be found; but, as will turn out, the source function is not unique. The ambiguity is caused by the dependence of the source on the two independent variables \( x \) and \( t \). Once the dependence on one variable is prescribed, for instance a localized force that acts only at the point \( x=0 \), the source will be uniquely defined by the signal. The ambiguity can be exploited to satisfy additional requirements, as will become evident in the next subsection.

To obtain the condition for the source, consider the temporal Fourier transform of Eq. (5), which reads

\[
( -i \omega + i \omega \gamma_x) \mathcal{S}(k, \omega) = \mathcal{S}_1(k, \omega)
\]

For \( \mathcal{S}_1 = 0 \) this requires that the dispersion relation \( \omega = \Omega_1(k) \) should be satisfied. The condition for \( \mathcal{S} \)

\[
\mathcal{S}(k, \omega) = \frac{\mathcal{S}_1(k, \omega)}{i \Omega_1(k)}
\]

reads in physical space

\[
\eta(x, t) = \int \frac{\mathcal{S}_1(k, \omega)}{i \Omega_1(k)} e^{ikx - \omega t} dk d\omega
\]

which, specified for \( x=0 \), produces a condition for the source

\[
\mathcal{S}(t) = \int \frac{\mathcal{S}_1(k, \omega)}{i \Omega_1(k)} e^{ikx - \omega t} dk d\omega
\]

or equivalently

\[
\mathcal{S}(t) = \int \frac{\mathcal{S}_1(k, \omega)}{i \Omega_1(k)} dk
\]

Using the fact that the dispersion relation is invertible, a change of variables is made from \( k \) to \( \nu \) with \( \nu = \Omega_1(k) \). With the notation for the group velocity \( V_g(k) \) and the inverse \( K_1(\nu) \) such that \( \nu = \Omega_1(K_1(\nu)) \), it follows that \( d\nu = V_g(K_1(\nu))dk \), and hence

\[
\mathcal{S}(\omega) = \int \frac{\mathcal{S}_1(K_1(\nu), \omega)}{V_g(K_1(\nu))} \frac{d\nu}{i(\nu - \omega)}
\]

Assuming that \( \mathcal{S}_1(K_1(\nu), \omega)/V_g(K_1(\nu)) \) is an analytic function in the complex \( \nu \)-plane, Cauchy’s principal value theorem leads to the result that

\[
\mathcal{S}(\omega) = 2\pi \frac{\mathcal{S}_1(K_1(\omega), \omega)}{V_g(K_1(\omega))}
\]

and hence

\[
\mathcal{S}_1(K_1(\omega), \omega) = \frac{1}{2\pi} V_g(K_1(\omega)) \mathcal{S}(\omega)
\]

This is the source condition, the condition that \( \mathcal{S}_1 \) produces the desired elevation \( \eta(t) \) at \( x=0 \). This condition shows that the function \( \omega \rightarrow \mathcal{S}_1(K_1(\omega), \omega) \) is uniquely determined by the given time signal. However, the function \( \mathcal{S}_1(k, \omega) \) of 2 independent variables is not uniquely determined; it is only uniquely defined for points \( (k, \omega) \) that satisfy the dispersion relation. Consequently, the source function \( \mathcal{S}_1(x, t) \) is not uniquely defined, and the spatial dependence can be changed when combined with specific changes in the time dependence.

To illustrate this, and to obtain some typical and practical results, consider sources of the form

\[
\mathcal{S}_1(x, t) = \text{g}(x)f(t)
\]

in which space and time are separated; \( g \) describes the spatial extent of the source, and \( f \) is the so-called modified influx signal. Then \( \mathcal{S}_1(k, \omega) = \hat{g}(k)f(\omega) \) and the source condition for the functions \( f \) and \( g \) together is written as

\[
\hat{g}(K_1(\omega))f(\omega) = \frac{1}{2\pi} V_g(K_1(\omega)) \mathcal{S}(\omega)
\]

Clearly, the functions \( f \) and \( g \) are not unique, which is illustrated for two special cases.

Point A source that is concentrated at \( x = 0 \) can be obtained generation: using the Dirac delta-function \( \delta_{\text{Dirac}}(x) \). Then taking \( \mathcal{S}_1(x, t) = \delta_{\text{Dirac}}(x)f(t) \), it follows (using \( \delta_{\text{Dirac}}(k) = 1/(2\pi) \)) that \( \mathcal{S}_1(k, \omega) = f(\omega)/2\pi \). The source condition then specifies the modified influx signal \( f(t) \)

\[
\mathcal{S}_1(x, t) = \delta_{\text{Dirac}}(x)f(t) \quad \text{with} \quad f(\omega) = V_g(K_1(\omega))\mathcal{S}(\omega)
\]

Observe that in physical space, the modified signal \( f(t) \) is the convolution between the original signal \( \eta(t) \) and the inverse temporal Fourier transform of the group velocity \( \omega \rightarrow V_g(K_1(\nu)) \).

Area A more general choice of the spatial extent of the extended source, given by a function \( g(x) \), determines the influx generation: signal according to

\[
\hat{f}(\omega) = \frac{1}{2\pi} \frac{V_g(K_1(\omega))}{\hat{g}(K_1(\omega))} \mathcal{S}(\omega)
\]

In particular, it is possible to influx the original signal i.e. \( f(t) = \eta(t) \) provided we choose \( \hat{g}(k) = (1/2\pi)V_g(k) \), so that

\[
\mathcal{S}_1(x, t) = \eta(x)\mathcal{S}(t)
\]

where \( \eta(x) \) has been introduced above (2). In view of the scaling properties, the extent will become large for deep water, which may not be a desirable choice. A smooth alternative would be to take a Gaussian profile.
such as \( g(x) = \exp(-x^2/\beta) \) where the parameter \( \beta \) can control the practical extent of the source area, as has been used by Wei et al. (1999).

As a final remark, notice that the area extended and the point generation are the same for the case of the non-dispersive shallow water limit for which \( \Omega_s(k) = C_0 k \) and \( V_g(k) = C_0 \) (which then coincides with the phase velocity). In that case \( S_f(K(\omega), \omega) = C_0 \sqrt{s(\omega)/2\pi} \) and the familiar result for inflow of a signal \( s(t) \) at \( x = 0 \) is obtained

\[
\partial_t \eta = -C_0 \partial_x \eta + C_0 \delta_{\text{Drac}}(x)s(t)
\]

2.3. Inflowing in bi-directional equations

For the uni-directional equations in the previous subsection the solution is uniquely determined by the specification of the elevation at one point. For bi-directional equations \((\partial_t^2 + D)\eta = 0\) this is obviously no longer the case, since the two propagation directions have to be distinguished. Hence, the inflowing from one point \( x = 0 \) will need the signals \( s_{l} \) and \( s_r \) to specify the right and left travelling wave respectively.

Since a sum of sources leads to the sum of the generated waves, it is sufficient to construct uni-directional influx sources, i.e. to determine for given signal \( s_0(t) \) the source \( H(x, t) \) so that

\[
(\partial_t^2 + D)\eta = H(x, t)
\]

has solution \( \eta \) such that \( \eta(x, t) = 0 \) for \( x < 0 \) and \( \eta(x, t) \) is the wave travelling to the right with signal \( s_0(t) \) at \( x = 0 \).

Let \( S(x, t) = g(x)f(t) \) be a source in the to the right running equation with signal \( s_0(t) \) at \( x = 0 \) and let \( \eta_r \) be the solution (vanishing for \( x < 0 \))

\[
(\partial_t + A_1)\eta_r(x, t) = S(x, t)
\]

Then applying the operator \((\partial_t - A_1)\) to this equation it follows that \( \eta_r \) satisfies

\[
(\partial_t^2 + D)\eta_r = (\partial_t - A_1)S(x, t) = g(x)f(t) - f(t).A_1g(x)
\]

For the case that \( g \) is an even function of \( x \), it follows that this forced equation only produces the desired solution \( \eta_r \). Indeed, since the part \( g(x)f(t) \) in the source will produce an even function, the symmetrization of \( \eta_r \), while the odd part \(-f(t).A_1g(x)\) will produce the skew-symmetrization of \( \eta_r \), the sum of the sources produces the sum of the symmetrization and the skew-symmetrization, which is \( \eta_r \). Hence, if \( S_r = g(x)f(t) \) with \( g \) symmetric satisfies the uni-directional source condition \( \dot{g}(K(\omega))\dot{f}(\omega) = V_g(K_1(\omega))s_0(\omega)/(2\pi) \) then

\[
H(x, t) = (\partial_t - A_1)[g(x)f(t)]
\]

As a simple example, for the shallow water equation with uni-directional point source \((\partial_t + C_0\partial_x)\eta = C_0 \delta_{\text{Drac}}(x)s_0(t)\), the uni-directional inflowing to the right in the second order equation is given by

\[
(\partial_t^2 - C_0^2 \partial_x^2)\eta = (\partial_t - C_0 \partial_x)[C_0 \delta_{\text{Drac}}(x)s_0(t)]
\]

\[
= C_0 \delta_{\text{Drac}}(x)s_0(t) - C_0^2 \delta_{\text{Drac}}(x)s_0(t)
\]

with \( \delta_{\text{Drac}} \) being the derivative of Dirac’s delta function.

2.4. Equations in Hamiltonian form

Many Boussinesq-type of models are not formulated as a second order in time equation but rather as a system of two first order equations. As an example, the formulation that is closest to the basic physical laws uses the elevation \( \eta \) and the fluid potential at the surface \( \phi \) as basic variables. The governing equation is of Hamiltonian form and reads

\[
\partial_t \eta = \frac{1}{g} \partial_x \phi, \quad \partial_t \phi = -g \eta
\]

The first equation is the continuity equation, and the second the Bernoulli equation. Note that by eliminating \( \phi \), the second order equation \( \partial_t^2 \eta = -D \eta \) of the previous subsection is obtained.

The Hamiltonian structure is recognized for the Hamiltonian

\[
\mathcal{H} = \frac{1}{2} \int (g\eta^2 + \frac{1}{g} \partial_x \phi \partial_t \phi) \, dx = \frac{1}{2} \int (\eta^2 + \frac{1}{4}A_1 \phi^2) \, dx
\]

which has variational derivatives \( \delta \mathcal{H}/\delta \eta = g \eta \) and \( \delta \mathcal{H}/\delta \phi = D \phi/g \), so that the system is indeed in canonical Hamiltonian form:

\[
\partial_t \eta = \delta \mathcal{H}/\delta \phi, \quad \partial_t \phi = -\delta \mathcal{H}/\delta \eta
\]

For the formulation with \( \eta, \phi \), consider the forced equations

\[
\begin{cases}
\partial_t \eta = \frac{1}{g} \partial_x \phi + G_1 \\
\partial_t \phi = -g \eta + G_2
\end{cases}
\]

In the following only the special cases of elevation inflowing, i.e. taking \( G_2 = 0 \), and velocity inflowing for which \( G_1 = 0 \) will be considered.

2.4.1. Elevation inflowing

With \( G_2 = 0 \), upon eliminating \( \phi \) the equation becomes

\[
\partial_t^2 \eta = -D \eta + \partial_t A_1 \eta
\]

This is the same as the forced second order equation (12) of the previous subsection. Hence if \( H \) is the source (13) for uni-directional inflowing, \( G_1 \) is obtained from

\[
\partial_t G_1 = H = (\partial_t - A_1)[g(x)f(t)]
\]

or

\[
G_1 = g(x)f(t) - (\partial_t - A_1)f(t).A_1g(x)
\]

Note the (skew-) symmetry of the two contributions to the source as before.

2.4.2. Velocity inflowing

In many Boussinesq formulations the velocity is used instead of the potential. Writing \( u = \partial_t \phi \) the equations with forcing in the velocity become

\[
\begin{cases}
\partial_t \eta = \frac{1}{g} \partial_x c^2 u \\
\partial_t u = -g \partial_x \eta + G_3
\end{cases}
\]

where \( c^2 = -D/k^2 \) is the squared phase velocity operator. By eliminating \( \eta \) the second order equation for \( u \) is obtained

\[
\partial_t^2 u = -D u + \partial_t A_1 \eta
\]

This is the same as Eq. (15) for the uni-directional elevation inflowing, and \( G_3 \) can be specified if the velocity at \( x = 0 \) is given, say \( u(0, t) = s_1(t) \)

\[
G_3 = g(x)f(t) - (\partial_t - A_1)f(t).A_1g(x)
\]

and

\[
\dot{g}(K_1(\omega))\dot{f}(\omega) = \frac{1}{2\pi} V_g(K_1(\omega))s_1(\omega)
\]

3. Multi-directional wave equations

In this section the results of the previous section are generalized to multi-directional waves in 2D in a straightforward way. The notation for the horizontal coordinates is \( \mathbf{x} = (x, y) \) and for the wave vector \( \mathbf{k} = (k_x, k_y) \); the lengths of these vectors are...
written as \( x = \mathbf{x} \) and \( k = |\mathbf{k}| = \sqrt{k_x^2 + k_y^2} \) respectively. In 2D the spatial transform is

\[
\eta(\mathbf{x}) = \int \eta(\mathbf{k}) e^{i \mathbf{k} \cdot \mathbf{x}} \, d\mathbf{k}, \quad \tilde{\eta}(\mathbf{k}) = \frac{1}{(2\pi)^2} \int \eta(\mathbf{x}) e^{-i \mathbf{k} \cdot \mathbf{x}} \, d\mathbf{x}
\]

The dispersion relation is the relation between the wave vector \( \mathbf{k} \) and the frequency \( \omega \) so that the plane waves \( \exp(i(\mathbf{k} \cdot \mathbf{x} - \omega t)) \) are physical solutions.

### 3.1. Forward propagating waves

In 2D a skew-symmetric operator \( \mathcal{A}_e \) will be defined for given direction vector \( \mathbf{e} \) to formulate first order dynamic equations that describe forward or backward wave propagation with respect to the vector \( \mathbf{e} \). Forward propagating wave modes have a wave vector that lies in the positive half-space \( \{ |\mathbf{k}| |\mathbf{k}.\mathbf{e}| > 0 \} \) while the wave vector of backward propagating modes lies in the negative half-space \( \{ |\mathbf{k}| |\mathbf{k}.\mathbf{e}| < 0 \} \). First order in time equations for forward or backward travelling waves is most useful for wave inflow in a specific part of a half plane, for instance when waves are generated in a hydrodynamic laboratory, or when dealing with coastal waves from the deep ocean towards the shore. The embedded forcing in the first order equations will also help us to determine the forcings in second order in time multi-directional equations.

The first order in time equations are obtained with an operator \( \mathcal{A}_e \) that is defined as the pseudo-differential operator that acts in Fourier space as multiplication as

\[
\mathcal{A}_e \delta_2(k) = \text{sign}(\mathbf{k}) \mathbf{e} \delta_2(k)
\]

As before \( \omega^2 = \Omega(k)^2 = D(k) \), but now \( k = |\mathbf{k}| \) in \( \Omega(k) \) only takes nonnegative values. Since \( \delta_2(k) = -\delta_2(-k) \) the operator \( \mathcal{A}_e \) is real and skew-symmetric. Observe that \( \delta_2 \) has discontinuity along the direction \( \mathbf{e}^- \) (perpendicular to \( \mathbf{e} \)). The 2D forward propagating dispersive wave equation is then given as

\[
\partial_t \zeta = -\mathcal{A}_e \zeta
\]

which has as basic solutions the plane waves \( \exp(i(\mathbf{k} \cdot \mathbf{x} - \Omega(k)t)) \). Without restriction of generality we will take in the following \( \mathbf{e} = (1, 0) \) so that \( \delta_2 = \text{sign}(k_x) \Omega_2(k) \).

For the 2D excitation problem with embedded forcing

\[
\partial_t \tilde{s} = -\mathcal{A}_e \tilde{s} + S(\mathbf{x}, t)
\]

consider inflowing from the \( y \)-axis into the half space with \( x > 0 \), such that the source \( S(\mathbf{x}, t) \) has to be determined such that at the \( y \)-axis

\[
\eta(0, y, t) = s(y, t)
\]

for a prescribed signal \( s(y, t) \).

Applying the same technique as in the 1D case, we obtain that \( S(\mathbf{x}, t) \) has to satisfy the source condition

\[
s(y, t) = \frac{\mathcal{S}(k_x, k_y, \omega)}{\Omega_2(k_x, k_y, \omega)} e^{i k_x y + i \omega t} \, dk_x \, dk_y \, d\omega
\]

or equivalently

\[
\tilde{s}(k_x, \omega) = \frac{\mathcal{S}(k_x, k_y, \omega)}{\Omega_2(k_x, k_y, \omega)} \, dk_y
\]

Now a change of integration variable is made from \( k_x \) to \( \nu = \Omega_2(k_x, k_y) \), which is possible because of the monotonicity of \( \Omega_2 \) with respect to \( k_x \) at fixed \( k_y \) leading to \( k_x = K_\nu(k_y, \nu) \). Writing \( K_\nu(k_y, \nu) = \sqrt{K_x^2 + K_y^2} \) and using

\[
\frac{d\nu}{dk_x} = \text{sign}(k_x) \delta_2 \frac{dk}{dk_x} = \frac{V_g(k_y)}{V_g(k_y)}
\]

there results

\[
\tilde{s}(k_y, \omega) = \frac{\mathcal{S}(K_\nu(k_y, \nu), k_y, \omega)}{K_\nu(k_y, \nu)} \frac{K_\nu(k_y, \nu)}{V_g(K_\nu(k_y, \nu))} \, dk
\]

With Cauchy’s integral theorem the source condition in 2D is obtained as

\[
\mathcal{S}(k_x(k_y, \omega), k_y, \omega) = \frac{1}{2\pi i} V_g(K(k_y, \omega)) \frac{\tilde{s}(k_y, \omega)}{K(k_y, \omega)}
\]

Just as in 1D, note that the source \( S \) in not unique: \( \mathcal{S}(k_x, k_y, \omega) \) is unique only on the 2-dimensional subspace for which \( k_x = K_\nu(k_y, \omega) \).

For separated sources of the form

\[
S(x, y, t) = g(x) f(y, t)
\]

it follows that \( \mathcal{S}(k_x, k_y, \omega) = \tilde{g}(k_y) \tilde{f}(k_x) \). Hence, for a given function \( g(x) \), the function \( f(y, t) \) should be chosen as the inverse Fourier transform of \( \tilde{f}(k_x, \omega) \) with

\[
\tilde{f}(k_x, \omega) = \frac{1}{2\pi} \frac{V_g(K(k_y, \omega)) |K_\nu(k_y, \omega)|}{k_x} \tilde{s}(k_y, \omega)
\]

Some characteristic special cases are considered below.

**Uniform** Horizontal inflowing from the \( y \)-axis is described by horizontal specifying the same signal at each point: inflowing \( s(y, t) = s_1(t) \). Hence \( \tilde{s}(k_y, \omega) = \delta_{\text{Dirac}}(k_y) s_1(\omega) \) and this leads to

\[
\tilde{f}(k_x, \omega) = \frac{\delta_{\text{Dirac}}(k_x)}{2\pi} \frac{s_1(\omega)}{|K(k_y, \omega)|} \frac{V_g(K(k_y, \omega))}{K(k_x, \omega)} \tilde{s}(k_y, \omega)
\]

Since now \( |K(k_y, \omega)| = K(k_y, \omega) \) and \( K_\nu(k_y, \omega) = K_\nu(k_y, \omega) \) with \( k_0 \) as introduced above, we get

\[
\tilde{f}(k_x, \omega) = \frac{\delta_{\text{Dirac}}(k_x)}{2\pi} \frac{s_1(\omega)}{|K(k_y, \omega)|} \frac{V_g(K(k_y, \omega))}{K(k_x, \omega)} \tilde{s}(k_y, \omega)
\]

which is the result as can be expected from the 1D case, Eq. (11).

**Oblique** Waves being inflow from the \( y \)-axis are determined wave by a specified signal \( s(y, t) \). The spatial–temporal generation Fourier transform \( \tilde{s}(k_x, k_y, \omega) \) determines the amplitude of the plane wave with wave vector \( \mathbf{k} = (k_x, k_y) = k(\cos \theta, \sin \theta) \), where \( k \) is determined from \( \omega \) through the dispersion relation, and then \( \theta \) is found from \( k_y \), so that \( \tilde{s}(k_x, \omega) \) also specifies the wave direction for each frequency. For instance, inflowing waves under a fixed angle \( \theta_0 \) correspond to \( \tilde{s}(k_x, \omega) = e^{i (\omega \sin \theta_0)} \delta(k_y - K_\nu(\omega) \sin \theta_0) \), i.e. \( s(y, t) = f(\omega) \exp[iK_\nu(k_y) \sin(\omega \sin \theta_0)] \, d\omega \) for a single frequency \( \omega_0 \) with corresponding \( k_y^0 = k_0 \sin \theta_0 \), \( k_0 = K_\nu(\omega_0) \), the source condition leads to the influx function (20)

\[
\tilde{f}(k_x, \omega) = \frac{1}{2\pi} \delta(k_y - k_0^0) s_1(\omega_0) \frac{V_g(K_\nu(\omega_0))}{|K(k_y, \omega)|} \frac{K_\nu(k_y, \omega)}{K(k_x, \omega)} \tilde{s}(k_y, \omega)
\]

Transforming to physical space, the source function is then found for wave inflow of amplitude \( a \)

\[
S(x, y, t) = g(x) a e^{i K_\nu(y - a_0) \frac{V_g(K_\nu(\omega_0))}{2\pi} \tilde{f}(k_y, \omega) \, \cos(\theta_0)}
\]

The above result is a generalization of well known results in the literature. For the specific choice \( g(x) = \exp(-\beta x^2) \) the last result is the same as the forcing derived by Wei et al. (1999). For \( g(x) = \delta_{\text{Dirac}}(x) \) this forcing has been used by Kim et al. (2007):

\[
S(x, y, t) = \delta_{\text{Dirac}}(x) a e^{i K_\nu(y - a_0) \frac{V_g(K_\nu(\omega_0))}{2\pi} \tilde{s}(k_y, \omega) \cos(\theta_0)}
\]

### 4. Adjustment scheme for nonlinear influxing

The source functions for influxing waves introduced in the previous sections were derived for linear evolution equations.
The sources turn out to be accurate for such linear models, and to a lesser extent to generate mild waves in weakly nonlinear models. To generate highly nonlinear waves with linear generation methods, one adjustment will be described here. For shortness, the description is restricted to multi-directional dispersive wave equation, but the scheme can also be applied for forward propagating equations.

When nonlinear waves are generated with the linear sources, undesirable spurious free waves will be generated. This problem is well known from wavemaker theory; much research has been devoted to model second and third order wave steering for flap motion, see e.g. Schäffer (1996), van Leeuwen and Klopman (1996), Schäffer and Steenberg (2003) and Henderson et al. (2006). Fuhrman and Madsen (2006) studied short-crested wave simulations that result in characteristic hexagonal and rectangular wave forms, inspired by the physical experiments of Hammack et al. (2005). In these physical experiments of Hammack et al., as well as in the numerical simulation of Fuhrman and Madsen (2006), a linear wavemaker method was used to generate the (nonlinear) short-crested waves. The nonlinear model, and the physical experiment, responded by releasing spurious free harmonics due to the fact that third-order components in the wave generation are neglected. This resulted in modulations in the computational domain and in the physical experiment. Fuhrman and Madsen showed that inclusion of the third-order wave components in the wave generation reduces significantly the first-harmonic spurious modulations. This shows that wavemaker theory should take higher order harmonic steering into account when dealing with highly nonlinear waves.

The appearance of spurious free waves can also be expected in embedded wave generation methods if the force function is derived for a linear(ized) wave model. Wei and Kirby (1998) used embedded wave generation methods if the force function is when dealing with highly nonlinear waves.

Theory should take higher order harmonic steering into account due to the fact that third-order components in the wave (nonlinear) short-crested waves. The nonlinear model, and the numerical simulation of Fuhrman and Madsen (2005). In these physical experiments of Hammack et al., as well as in the numerical simulation of Fuhrman and Madsen (2006), a linear wavemaker method was used to generate the (nonlinear) short-crested waves. The nonlinear model, and the physical experiment, responded by releasing spurious free harmonics due to the fact that third-order components in the wave generation are neglected. This resulted in modulations in the computational domain and in the physical experiment. Fuhrman and Madsen showed that inclusion of the third-order wave components in the wave generation reduces significantly the first-harmonic spurious modulations. This shows that wavemaker theory should take higher order harmonic steering into account when dealing with highly nonlinear waves.

The appearance of spurious free waves can also be expected in embedded wave generation methods if the force function is derived for a linear(ized) wave model. Wei and Kirby (1998) used a numerical filtering method proposed by Shapiro (1970) in order to reduce the effects of the spurious free waves. They conclude that the method is cumbersome to write and inconvenient to code in the program.

Instead of using higher order steering or numerical filtering, we propose to use an adjustment for nonlinear wave generation that is motivated by Dommermuth (2000). Dommermuth remarked that nonlinear dispersive wave models can be initialized with linear wave fields if the flow field is given sufficient time to adjust. For the initial value problem which he investigated, he introduced an adjustment scheme in time that allows the natural development of nonlinear self-wave (locked modes) and wave-wave (free modes) interactions. To implement this idea in nonlinear wave models, the higher-order terms, denoted by \( F \), are multiplied by a slowly increasing function from 0 to 1 in a time interval \( T_w \), leading to the adjustment \( \hat{F} \) given by

\[
\hat{F} = (1 - \exp(-(t/T_w)^n))F
\]

for some positive power \( n \). In his examples, the optimal length of the time interval \( T_w \) should be larger than two times the period of the longest waves in the simulation.

For embedded wave generation, which takes place in time during the whole simulation, we modify the adjustment accordingly: the influxed waves are propagated away from the influx position by a spatially dependent increase of the nonlinear terms of the equation. Specifically, consider embedded influxing in a nonlinear Hamiltonian model with force functions (14) and with additional nonlinear (higher order) terms \( N_1 \) and \( N_2 \), given by

\[
\frac{\partial \eta}{\partial t} = \frac{D}{g} \phi + G_1 + N_1
\]

\[
\frac{\partial \phi}{\partial t} = -g \eta + G_2 + N_2
\]

The adjustment scheme in space uses a characteristic function \( \chi(x, L_a) \) that gradually grows from 0 to 1 in a transition zone with length \( L_a \); multiplying the nonlinear terms to \( N_1 \) and \( N_2 \) with this function results in

\[
\frac{\partial \eta}{\partial t} - \frac{D}{g} \phi - G_1 = \chi N_1
\]

\[
\frac{\partial \phi}{\partial t} + g \eta - G_2 = \chi N_2
\]

Fig. 3 illustrates the characteristic function \( \chi(x, L_a) \) which starts at the influx point \( x = 0 \) and increases in the propagation direction. For illustration, we perform several numerical simulations with the nonlinear Variational Boussinesq Model (Adytia and van Groesen, 2012), to test and validate the method.

The simulations aim to generate harmonic waves with period \( 5 \) s in a numerical basin with a depth of \( 2 \) m and length \( 15 L \), where \( L \) is the wavelength. The waves are generated at \( x = 0 \) with the (bidirectional) elevation inflowing. At both ends of the basin, sponge-layers are placed to damp the waves. To test the adjustment-scheme, and the required length of the adjustment interval, various values of the amplitude are considered, corresponding to wave steepness in between \( ka = 0.0075 \) and \( ka = 0.12 \). In Fig. 4 simulations with the linear model are shown in the first row, and simulations with the nonlinear model without and with adjustment in the second and third row respectively. The appearance of spurious free waves is clearly pronounced when the nonlinear simulation is performed without the adjustment scheme. By using the length of the adjustment interval according to the information in Table 1, the results with the fully nonlinear VBM give good agreement with the 5th order Stokes waves (Fenton, 1985) as illustrated in Fig. 5. A relative error of 2% compared to the 5th order Stokes waves has been used to determine the minimal length of the adjustment interval.

To analyze the resulting harmonic evolution in more detail, a Fast Fourier Transform (FFT) analysis of the time series at each computational grid point has been performed. Fig. 6 shows the first-order (solid line) and the second-order (dotted line) amplitudes for various simulation methods: with the linear code (upper left plot), with the nonlinear code without adjustment (upper right plot), and with an adjustment interval of \( 2L \) (lower left plot) and \( 5L \) (lower right plot). Since a linear influxing method misses the bound (second and higher) harmonics, a direct influx in the nonlinear model shows the release of spurious waves that compensate the missing bound waves. These spurious waves travel as free wave components, with opposite phase compared to the missing bound harmonic components in the linear influx signal (see also Fuhrman and Madsen, 2006). By applying an adjustment interval of sufficient length, shown in the lower right plot of Fig. 6, the second harmonic grows slowly to nearly steady in the adjustment zone, taking some energy from the first harmonic. If the length of the adjustment zone is not sufficiently long, for instance
method is still somewhat ad hoc and further investigations are desired.

As a final remark it is noted that the adjustment scheme can also be applied in such a way as to retain the Hamiltonian structure of the equations by multiplying the non-quadratic terms in the Hamiltonian by the adjustment function. Comparison of simulations with both methods did not show noticeable differences.

### 5. Numerical simulations

In this section, the performance of the embedded influx methods are illustrated with simulations of two numerical codes. One code is a spectral implementation of the equations with exact dispersion. Results of simulations will be shown that are obtained with AB-models that have exact linear dispersion and are accurate up to and including second order terms; see van Groesen and Andonowati (2007), van Groesen et al. (2010), and van Groesen and van der Kroon (2012) for the 1D and She Liam and van Groesen (2010) for the 2D model.

The other code is based on the Variational Boussinesq Model which has approximate dispersion as described in Section 2; see Klopman et al. (2010), Lakhturov et al. (2012), and Adytia and van Groesen (2012). To use the embedded influxing method in the FE implementation of this Model, the source functions have to be constructed using the dispersion relation of the VBM itself; after transformation to physical space, the sources have to be discretized in the FE setting.

#### 5.1. 1D nonlinear wave focusing

For a case of strong nonlinear wave focusing, simulations with embedded point generation in the nonlinear AB equation are compared with experiments. The measurements were done at MARIN hydrodynamic laboratory (Maritime Research Institute Netherlands), case 109001. In a long tank with depth of 1 m, the time signal of the measured surface elevation at one position, say at $x=0$, is taken as the influx signal, and measurements at two other positions $x=19.2$ m and $x=20.8$ m are used for comparison. The influxed signal consists of short waves followed by longer waves that have faster speed. The broad spectrum, and the strong focusing effect (with more than threefold amplitude amplification compared to the maximal influx amplitudes) make this a suitable test for the influx performance.

The plots of the influx signal, and the modified signal that is used in the source term, are shown side by side in the first row of Fig. 7, with the spectra of the two signals below it. Notice that the modified signal has higher amplitude and spectrum because of the multiplication with the group velocity as in expression (10). The comparison of results of the numerical simulation with the measurements is shown in Fig. 8 at two positions, one close-by and the other at almost the exact position of focusing. This figure shows that the focusing phenomenon, longer waves catch up with shorter waves and interfere constructively at the focusing point, is not only qualitatively but also quantitatively well-captured by the simulation.

#### 5.2. Oblique wave interaction

To illustrate influxing of oblique plane waves, an example is considered of oblique wave interaction in MARIN measurements in a wide tank of 5 m depth for 300 s. One wave is influxed from the $y$-axis for $y \in [10, 27]$ parallel to the $x$-axis and has a period of 1.8 s and an amplitude of 0.1 m. The second wave is influxed from the $x$-axis for $x \in [11, 150]$ and has period 2.2 s, amplitude 0.1 m and makes an angle of 30° with the positive $x$-axis.
Simulation of the nonlinear bidirectional biharmonic waves is done with inflowing for individual flap motion using the source term given by (21) in the nonlinear AB2-spectral code. The simulated elevation is shown in the density plot of Fig. 9 at time \( t = 300 \) s; the time signals at one position are compared with measurements for each individual wave and for the two waves together. The interaction shows the characteristic pattern of oblique bichromatic waves with small nonlinear effects.

### 5.3. Forward propagating inflowing

1D simulations with the finite element VBM code are performed to illustrate six different inflowing methods. Elevation and velocity inflowing is used to generate symmetric or skew-symmetric bi-directional waves or to produce only forward propagation waves. Area inflowing is used with taking for the spatial function in the sources (11) the function \( \gamma(x) \) related to the group velocity in Fourier space (2). The six simulations are done for 60 s on 1 m water depth. The computational domain is from \( x = -50 \) m until \( x = 50 \) m with the wave generation at the origin.

The signal to be inflowed is chosen to be a bipolar given by

\[
\eta_0(t) = 0.2(t - 30)\exp(- (t - 30)^2)
\]

The corresponding initial signal for the velocity inflowing is found from \( u_0(t) = iK_1(\omega)\phi_0 \) with \( \phi_0 = (-i\gamma\eta_0(\omega)/\omega) \). Fig. 10 shows plots of the simulation results for the wave profile at time 40 s; both elevation and velocity generation give the same result as expected.

### 6. Conclusion

In a rather straightforward way source functions have been derived that are added to first and second order time equations of Boussinesq type to generate desired wave fields. It was shown that the source functions are not unique, but that the temporal–spatial Fourier transform is unique when the dispersion relation is satisfied. This ambiguity of the source function has been exploited to reduce or enlarge the extent of the generation area. Inflowing from a point or line requires the modified signal to be higher, due to the multiplication in temporal Fourier space with the group velocity of the desired inflow signal; for generation areas of larger extent, the modified signal is lower, but the waves are only accurate outside the generation area.

Various test cases shown above illustrated the quality of wave generation by comparing with experimental data. The generation methods presented here were used in various other cases, such as simulations of irregular waves entering a harbour and simulation of bi-modal sea states consisting of swell and wind waves for research on predicting elevation at the position of a radar that scans the surrounding area with a nautical x-band radar. A report about nonlinear simulations for MARIN experiments of short crested waves is in preparation.

Being able to generate desired waves also gives the possibility to absorb them, partly or fully, or reverse their direction to simulate a fully or partly reflecting wall. The input signal is then determined by the incoming waves at the desired position.
active wave absorber, for instance, the opposite signal is generated and added to the incoming wave in the same propagation direction. If in addition a fraction of the signal is influxed in the opposite direction, a partly reflecting wall is obtained. In this way rather complex spatial geometries can be treated in a numerically accurate and efficient way.

Fig. 8. Part of the time signal in the first column and the amplitude spectrum in the second column of AB simulations (solid curve) and measurements (dashed curve). The first row is at position $x = 19.2$ m and the second row at position $x = 20.8$ m.

Fig. 9. The graphs in successive rows are showing the results of wave influxing along the $y$-axis, the $x$-axis and the simultaneous $x$- and $y$-axis. Shown on the left and on the right panel the plot of density and time signal at the position $x = 50$ m and $y = 15$ m after 100 s simulation.
Fig. 10. Snapshots of the wave profile at t = 40 s. The horizontal and the vertical axes are the space and the wave elevation in [m] respectively. The first column shows results of wave generation with elevation influxing while the second column shows results with velocity influxing. From the first to the last row are shown the results of the generation if a symmetric, a skew-symmetric and a uni-directional method is used.

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