Optimal relaxed causal sampling from system theoretic viewpoint

Hanumant Singh Shekhawat · Gjerrit Meinsma

Abstract This paper studies the design of an optimal relaxed causal sampler using sampled data system theory. A lifted frequency domain approach is used to obtain the existence conditions and the optimal sampler. A state space formulation of the results is also provided. The resulting optimal relaxed causal sampler is a cascade of a linear continuous time system followed by a generalized sampler and a discrete system.

1 Introduction

Most of the signals in real world are analog in nature (e.g. speech). To transmit these analog signals with high quality, these signals are often converted into discrete signals with the help of a sampler. At the receiving end, these discrete signals are converted back to the analog domain with the help of a hold. To measure the quality of these reconstructed signals, the sampled-data system theory can be used (see [26] and the references therein). A general sampled-data setup is shown in Figure 1. Here an analog signal $y$ is sampled by a sampler $S$, at sampling period $h$, to produce a discrete signal $\bar{y}$. Then the hold $H$ converts the discrete signal $\bar{y}$ back to the analog domain. This reconstructed signal $u$ must resemble our original analog signal $y$ (or $v$). To check the quality of the reconstruction process, $u$ is compared with signal $y$ (or $v$). A distinctive feature of sampled-data system theory is that analog signals $y$ and $v$ are modeled with a given linear continuous time system (LCTI) $G$ driven by a process $w$ with known characteristics. The quality of the reconstruction process is generally measured by $L^2$ (or $H^2$) or $L^\infty$ (or $H^\infty$) norms [2, 13] of the mapping from $w$ to $e = v - u$.

The sampled-data system theory was first applied in the signal reconstruction problem (i.e. to obtain the best reconstructed signal) in 1996 by Khargonekar and Yamamoto [6] (in 1995, Chen and Francis [25] used this theory to the signal reconstruction problem entirely in the discrete domain). Instead of aiming at exact reconstruction as in the Shannon case, minimization of the error without throwing away any frequencies is the main criterion in the signal reconstruction using sampled-data system theory. Starting from [6] in 1996, sampled data system theory
is applied to several signal processing applications using different error criteria (see e.g. [4, 22, 5, 14, 11, 24]). For a complete list of applications see the review paper by Yamamoto et al. [28].

Throughout this paper, we assume that the LCTI model $\mathcal{G}$ is known to us. Now, the design problem that can arise in practice may involve design of an optimal sampler $S$ given a hold $H$ or vice-versa. In this paper, we consider the problem of designing an optimal sampler $S$ given a hold $H$ and a signal generator $G$. A non-causal sampler can be obtained by using the solution provided in [14]. Ignoring causality is not a realistic because most of the systems in practice are causal or relaxed causal in nature. Relaxed causal systems loosely speaking are systems whose present output depends not only on the present and the past inputs but also on a limited set of the future of the input. In this paper, we concentrate on the design of relaxed-causal samplers instead of non-causal samplers, given causal hold $H$ and causal signal generator $G$. The hold $H$ in this paper are assumed causal because any finite relaxation in causality can be shifted to the sampler.

The dual problem of designing an optimal relaxed causal hold $H$ given a sampler is studied and solved by [21] and [12]. The design of an optimal causal zero order hold $H$ given an ideal sampler $S$ (or vice-versa) is also well known (see [18, 19, 21]). All of these papers use the lifting technique to achieve the goal (see [13] for a review of lifting and lifting transforms). We study the problem of designing optimal stable relaxed causal samplers $S$ given causal $H$ and $G$. This problem is similar to a two-sided model matching for LCTI systems (see [9]) but with the fundamental difference that systems involved here are required to be linear $h$-time shift invariant. Lifting also helps in obtaining a solution for this problem. We provide a frequency domain solution as well as a ready to use state space solution using the machinery given in [16].

The rest of the paper is organized in four sections. In sections 2-5, we briefly provide some fundamentals of the sampled-data system theory, lifting, lifting transforms, lifted transfer functions, lifted-causality and norms. In Section 6 we state our problem more precisely and provide a (lifted) frequency domain abstract solution. In Section 7, we review the fundamentals of the state-space for linear $h$-time shift invariant systems. In Section 8 a state space solution is provided to the problem.

Notation: $\mathbb{Z}$ is the set of integers and $\mathbb{N}$ is the set of non-negative integers. $\mathbb{C}$ denotes the set of complex number. The closed unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$ and the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. proj$_A$ is the orthogonal projection on the Hilbert space $A$. $L^2[0, h]$ is the space of square integrable functions defined on $[0, h]$. $C^2(X_1, X_2)$ is the space of square integrable continuous functions which map space $X_1$ to $X_2$. $\| \cdot \|_H$ denotes the Hilbert-Schmidt norm and $\| \cdot \|_\infty$ denotes the induced 2-norm of an operator. Linear discrete time invariant (LDTI) system means a linear $h$-time shift invariant system.

2 Sampled data setup

In this section, we will briefly review standard mathematical description of all components of sampled-data system given in Figure 1. For a detailed description, see [13]. The sampled data setup given in Figure 1 is redrawn in Figure 2 where $\mathcal{G} = \begin{bmatrix} \mathcal{G}_x \\ \mathcal{G}_y \end{bmatrix}$ is an LCTI (but not necessarily stable) system and $\mathcal{G}_x$ and $\mathcal{G}_y$ are the partition of $\mathcal{G}$ with respect to the signals $v$ and $y$ respectively.

![Fig. 2 Sampled-data setup](image)

The Sampler $S$ is a device which samples an analog signal $y : \mathbb{R} \to \mathbb{C}$ at every integer multiple of $h$ and gives a discrete signal $\bar{y} : \mathbb{Z} \to \mathbb{C}^r$. We assume $S$ to be linear and $h$-time shift invariant, i.e.

$$\bar{y} = Sy : \bar{y}[n] = \int_{-\infty}^{\infty} \psi(nh - s)y(s) \, ds.$$

where $\psi(t)$ is known as sampling function.

The hold $H$ is a device which converts a discrete signal $\bar{y} : \mathbb{Z} \to \mathbb{C}^r$ back to an analog signal $u : \mathbb{R} \to \mathbb{C}$. Here $r$ is a positive integer. We assume $H$ to be linear and $h$-time shift invariant, i.e.

$$u = H\bar{y} : u(t) = \sum_{n \in \mathbb{Z}} \phi(t - nh)\bar{y}[n], \ t \in \mathbb{R}.$$
The $\phi : \mathbb{R} \to \mathbb{C}^{1 \times r}$ is known as hold function.

3 lifting and transforms

To analyze, the analog and discrete signals in a common mathematical framework, we need concept of lifting. This section summarizes the lifting definition used in [13]. We also review the meaning of lifted transfer functions in this section. We start with lifting.

**Definition 1 (Lifting)** For a continuous time signal $f : \mathbb{R} \to \mathbb{C}^n$, the lifted signal $\tilde{f} : \mathbb{Z} \to \{[0, h) \to \mathbb{C}^n\}$ is the sequence of functions $\{\tilde{f}[k]\}$ defined as

$$\tilde{f}[k](\tau) := f(kh + \tau), \quad k \in \mathbb{Z}, \tau \in [0, h).$$

Here $\tau$ is inter-sample time parameter.

In a sense, we can say that lifting discretizes an analog signal. Note that lifting is an invertible process that means we never lose any intersample information due to the lifting. Lifting of a discrete signal results in the same signal therefore we will not use any special symbol for the lifted discrete signal.

The lifted $z$-transform of an analog signal $f$ is defined as

$$\tilde{f}(z; \tau) := \sum_{k \in \mathbb{Z}} \tilde{f}[k](\tau) z^{-k} = \sum_{k \in \mathbb{Z}} f[kh + \tau] z^{-k}. \quad (3)$$

3.1 Transfer function in lifted frequency domain

In this section, we define transfer functions for linear time invariant (LTI) systems, downsamplers and Holds in lifted frequency domain. For a more detailed discussion on (lifted) transfer function, we suggest [13].

Any linear system $G$ that is $h$-shift invariant in continuous time is by construction LTI in the lifted domain with respect to the discrete variable and hence can be written as a convolution

$$u = Gy \xrightarrow{\text{lift}} \tilde{u}[k] = \sum_i \tilde{G}[k - i]\tilde{y}[i].$$

Here $\tilde{G}[k - i]\tilde{y}[i]$ for each $i$ is a finite integral over inter-sample time [13]. Taking lifted Fourier transforms, we have

$$\tilde{u}(z) = \tilde{G}(z)\tilde{y}(z)$$

where $\tilde{G}(z) := \sum_i \tilde{G}[i]z^{-i}$ is known as the (lifted) transfer function of $G$.

Similarly, the (lifted) transfer function $\tilde{S}(z)$ of the sampler $S$ (defined in (1)) is given by [13]

$$\tilde{g}(z) := \tilde{S}(z)\tilde{y}(z) : \tilde{g}(z) = \int_0^h \tilde{\psi}(z; -\sigma)\tilde{y}(z; \sigma) d\sigma.. \quad (4)$$

Similarly, the (lifted) transfer function $\tilde{H}(z)$ of the Hold $H$ (defined in (2)) is given by [13]

$$\tilde{u}(z) := \tilde{H}(z)\tilde{y}(z) : \tilde{u}(z; \tau) = \tilde{\phi}(z; \tau)\tilde{y}(z). \quad (5)$$

3.2 Adjoint systems and conjugate transfer function

In this section we discuss adjoint and conjugate of a lifted system. It is shown in [13] that the kernel of adjoint $G^*$ of a linear system $G$ is $g^*(s, t) := (g(t, s))^*$, where * denote complex conjugate transpose and $g(t, s)$ is the kernel of $G$. Taking the lifted $z$-transform of the kernels, we have $\tilde{g}^*(z; \sigma, \tau) := \tilde{g}(1/z; \tau, \sigma)^*$. The system which has the kernel $g^*(z; \sigma, \tau)$ and it is known as the conjugate of the transfer function $G(z)$. It can be proved that for $z = e^{j\theta}$, the conjugate $G^*(e^{j\theta})$ is the adjoint of $G(e^{j\theta})$ with respect to $L^2[0, h]$ [13].

The kernel $\phi(t)$ of the adjoint $S^*$ of the sampler $S$ given in (1) is $\phi(t) := \psi(-t)^*$ and the kernel $\phi(z; \tau)$ of the conjugate $S^*(z)$ of the transfer function $\tilde{S}(z)$ given in (4) is $\phi(z; \tau) := \psi(1/z; -\tau)^*$ [13]. Similarly, the kernel $\psi(t)$ of the adjoint $H^*$ of the hold $H$ given in (2) is $\psi(t) := \phi(-t)^*$ and the kernel $\psi(z; \tau)$ of the conjugate $H^*(z)$ of the transfer function $\tilde{H}(z)$ given in (5) is $\psi(z; \tau) := \phi(1/z; \tau)^*$ [13]. Note that adjoint of sampler is a hold, and vice-versa.
4 Causality

Now, we define causality of the lifted systems. A shift invariant (lifted) system \( \tilde{G} \) (can be a lifted analog system \( \tilde{G} \) or discrete system \( \tilde{G} \) or lifted sampler \( \tilde{G} \) or lifted hold \( \tilde{G} \)) is defined (lifted) causal if
\[
\tilde{H}_k \tilde{G} (I - \tilde{H}_k) = 0, \quad \forall k \in \mathbb{Z}.
\]
where the truncation operator \( \tilde{H}_k \) is defined as
\[
(\tilde{H}_k \tilde{u})[n] := \begin{cases} \tilde{u}[n] & n < k \\ 0 & n \geq k \end{cases}.
\]
where the lifted \( \tilde{u} \) is \( \tilde{u} \) if the signal is analog or \( \bar{u} \) if the signal is discrete.

Remark 1 We can define causality of system in several ways. Not all definition of causality are equivalent to each other. For a detailed description of different definitions of causality and their relationship with each other, see [23, section 2.5] and [13]. In this paper, we consider only lifted causality of the systems. Therefore, from now on, whenever we refer to causality we mean lifted causality.

Also, we call a lifted signal \( \tilde{y} \) causal if \( \tilde{y}[n] = 0, \forall n < 0 \). Similarly, for a given integer \( l \), we call a lifted signal \( \tilde{y} \) \( l \)-causal if \( \tilde{y}[n+l] \) is causal.

For a given integer \( l \), a shift invariant lifted system \( \tilde{G} \) is defined \( l \)-causal or relaxed causal if
\[
\tilde{H}_{k-l} \tilde{G} (I - \tilde{H}_k) = 0, \quad k \in \mathbb{Z}
\]
This means that for a causal lifted input, the lifted output of an \( l \)-causal system is \( l \)-causal. In other words, the present lifted output at \( k \) depends upon all lifted inputs up to \( k + l \). Here, a causal system means \( l = 0 \) and a strictly causal system means \( l = -1 \). Anti-causality is just opposite of the causality. A system is defined anti-causal if its adjoint is causal. Note that the relaxed causal systems with \( l > 0 \) are neither causal nor anti-causal.

5 Lifted Norms and stability

In this section, we review the definition of some standard norms in sampled-data system theory. Norms defined in this section are very standard and discussed with great detail in [13,2,1]. In this section, system \( \mathcal{G} \) means all LDTI systems (including sampler, down sampler and hold). The (lifted) transfer function on the unit circle (in the complex plane) is denoted by \( \tilde{G}(e^{j\theta}) \).

1. \( L^2 \) is the space of LDTI systems with finite norm defined as [2,13]:
\[
\|\tilde{G}\|_{L^2} := \sqrt{\frac{1}{2\pi h} \int_{-\pi}^\pi \|\tilde{G}(e^{j\theta})\|_{HS} d\theta}
\]
where \( \|\cdot\|_{HS} \) stand for Hilbert-Schmidt norm of an operator [30, §8.1].

2. \( L^\infty \) is the space of LDTI systems with finite norm defined as [1,13]:
\[
\|\tilde{G}\|_{L^\infty} := \sup_{\theta \in [-\pi,\pi]} \|\tilde{G}(e^{j\theta})\|
\]
where \( \|\tilde{G}(e^{j\theta})\| = \sup_{x, \|x\|_2 = 1} \|\tilde{G}(e^{j\theta})x\|_2 \). Here \( \mathbb{S} \) means either \( L^2[0,h) \) or \( \mathbb{C}^n \) depending upon \( \bar{x} \) is lifted from an analog signal or discrete signal respectively.

3. The Hardy space \( H^p \) (\( p \) is either 2 or \( \infty \)) consists of those causal LDTI systems which are in \( L^p \) (see [13] for detail and more rigorous definition). For a given \( l \in \mathbb{N} \), \( z^lH^p \) is the space of LDTI systems with lifted transfer function \( U(z) \) such that the lifted transfer function \( z^{-l}\hat{U}(z) \) corresponds to a system in \( H^p \).

At any point of time, we never want that due to some bounded noise or external disturbances the output of our system grows unboundedly. Therefore, stability is desired for each component of the sampled-data setup given in Figure 2. In sampled-data system theory stability is defined as follows [13].

Definition 2 (Stability) An operator \( T : \mathbb{S}_i \rightarrow \mathbb{S}_o \), where \( \mathbb{S}_i \) and \( \mathbb{S}_o \) can be \( L^2(\mathbb{R}) \) or \( \ell^2(\mathbb{Z}) \), is stable if \( T \in L^\infty \).

So, stable means having a finite \( L^\infty \) norm.
6 Problem formulation and solution

In this section we formulate the problem of designing relaxed causal samplers given a hold and a model $\mathcal{G} := [\mathcal{G}_v \mathcal{G}_y]^T$. We also provide a (lifted) frequency domain solution of this problem. All the proofs of the results in this section are given in Appendix 11.1.

Now, we state our problem more precisely:

**Problem 1** Given causal $\mathcal{G}_v$ and $\mathcal{G}_y$, causal and stable hold $\mathcal{H}$, and $l \in \mathbb{N}$, find an $l$-causal and stable sampler $\mathcal{S}$ such that $\mathcal{G}_e:=\mathcal{G}_v - \mathcal{H}\mathcal{S}\mathcal{G}_y$ is stable and $\|\mathcal{G}_e\|_{L^2}$ is minimized. □

Intuitively, all the instabilities of $\mathcal{G}_v$ must be contained in $\mathcal{H}\mathcal{S}\mathcal{G}_y$ in order for $\mathcal{G}_e$ to be stable. As $\mathcal{H}$ and $\mathcal{S}$ are stable, this implies that $\mathcal{G}_y$ must be right invertible and stable. Moreover, $\mathcal{H}$ must pass all the instabilities. Therefore, the presence of a hold $\mathcal{H}$ complicates the question of existence of a solution of 1. The complexity of 1 is further increased as it is not immediately clear how the $l$-causality constraint can be imposed on the sampler. Similar to [21], lifting (and the lifted transform) can be used here to reduce some of these complexities. Hence, we write problem 1 in lifted z-domain as:

**Problem 2** Given $\tilde{\mathcal{G}}_v$ and $\tilde{\mathcal{G}}_y$ are causal, $\tilde{\mathcal{H}} \in \mathbb{C}^{\infty}$ and $l \in \mathbb{N}$, find $\tilde{\mathcal{S}} \in z^l\mathbb{C}^{\infty}$ such that $\tilde{\mathcal{G}}_e:=\tilde{\mathcal{G}}_v - \tilde{\mathcal{H}}\tilde{\mathcal{S}}\tilde{\mathcal{G}}_y \in L^\infty \cap L^2$ and $\|\tilde{\mathcal{G}}_e\|_{L^2}$ is minimized. □

Note that If there exists a solution of Problem 2 then $\tilde{\mathcal{G}}_e$ belongs to $z^l\mathbb{C}^{\infty}$. This is because $\tilde{\mathcal{G}}_v$, $\tilde{\mathcal{G}}_y$, $\tilde{\mathcal{H}}$ are causal and $\tilde{\mathcal{S}}$ is $l$-causal.

In order to solve 2, we break it into two parts according to the norm of $\tilde{\mathcal{G}}_e$:

1. **Stabilization problem:** find all $\tilde{\mathcal{S}} \in z^l\mathbb{C}^{\infty}$ such that $\tilde{\mathcal{G}}_e:=\tilde{\mathcal{G}}_v - \tilde{\mathcal{H}}\tilde{\mathcal{S}}\tilde{\mathcal{G}}_y \in L^\infty$.
2. **Optimization problem:** find an $\tilde{\mathcal{S}} \in z^l\mathbb{C}^{\infty}$ such that it solves the Stabilization problem and that minimizes $\|\tilde{\mathcal{G}}_e\|_{L^2}$.

First, we consider the stabilization problem and after parameterizing all its solutions, we consider the optimization problem. For existence and parameterization of all the solutions of the stabilization problem, we need the following assumptions:

**Assumption $A_1$:** $\mathcal{G}_v$ is rational, proper and causal.

**Assumption $A_2$:** There exists a factorization of $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_1\tilde{\mathcal{H}}_0$ with inner $\tilde{\mathcal{H}}_1 \in \mathbb{C}^{\infty}$ (i.e. $\tilde{\mathcal{H}}_1^*\tilde{\mathcal{H}}_1 = I$), and bistable and bicausal $\tilde{\mathcal{H}}_0 \in \mathbb{C}^{\infty}$. □

The factorization in Assumption $A_2$ is an example of inner-outer factorization of hold $\tilde{\mathcal{H}}$ (see [27, §6.3] for details).

Assumption $A_1$ guarantees the existence of a coprime factorization of $\tilde{\mathcal{G}}_y$ over $\mathbb{C}^{\infty}$ (see [27, theorem 4.2.4]). $\tilde{\mathcal{N}}_y$ and $\tilde{\mathcal{M}}_y$ are said to be left coprime factors in $\mathbb{C}^{\infty}$ of $\tilde{\mathcal{G}}_y$ if $\tilde{\mathcal{N}}_y$ and $\tilde{\mathcal{M}}_y$ are in $\mathbb{C}^{\infty}$, $\tilde{\mathcal{G}}_y = \tilde{\mathcal{M}}_y^{-1}\tilde{\mathcal{N}}_y$, and there exist Bezout factors $\tilde{\mathcal{X}}_1 \in \mathbb{C}^{\infty}$ and $\tilde{\mathcal{Y}}_1 \in \mathbb{C}^{\infty}$ such that

$$\tilde{\mathcal{M}}_y\tilde{\mathcal{X}}_1 + \tilde{\mathcal{N}}_y\tilde{\mathcal{Y}}_1 = I.$$ 

To have nice mathematical properties, the Holds considered in this paper are left invertible in $L^\infty$. Assumption $A_2$ implies left invertibility and stability of the hold. Assumption $A_2$ also helps in obtaining and parameterizing all the solutions of the stabilization problem as we will see later in this section. Both of assumptions $A_1$ and $A_2$ are used in the following proposition which states the condition of existence of solutions of the stabilization problem.

**Proposition 1** Given $\tilde{\mathcal{H}} \in \mathbb{C}^{\infty}$, causal $\tilde{\mathcal{G}}_v$ and $\tilde{\mathcal{G}}_y$, and $l \in \mathbb{N}$. If assumptions $A_1$ and $A_2$ are satisfied, then there exists a sampler $\tilde{\mathcal{S}} \in z^l\mathbb{C}^{\infty}$ such that $\tilde{\mathcal{G}}_e \in L^\infty$ iff the following three conditions hold:

1. $\Pi_{\tilde{\mathcal{H}}}\tilde{\mathcal{G}}_v \in L^\infty$ where $\Pi_{\tilde{\mathcal{H}}} := 1 - \tilde{\mathcal{H}}\tilde{\mathcal{H}}^*$,
2. there exists a coprime factorization over $\mathbb{C}^{\infty}$ of $\tilde{\mathcal{G}} := [\tilde{\mathcal{G}}_v \tilde{\mathcal{G}}_y]^T$ of the form

$$\tilde{\mathcal{G}} = \begin{bmatrix} I & \tilde{\mathcal{M}}_y \\ \tilde{\mathcal{M}}_y & 0 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\mathcal{N}}_v \\ \tilde{\mathcal{N}}_y \end{bmatrix}$$

with $\tilde{\mathcal{M}}_y, \tilde{\mathcal{N}}_y$ left coprime.
3. there exists a $\tilde{\mathcal{V}} \in L^\infty$ such that $\tilde{\mathcal{M}}_h := \tilde{\mathcal{H}}_{10}^{-1}\tilde{\mathcal{G}}_v - \tilde{\mathcal{V}}\tilde{\mathcal{M}}_y \in z^l\mathbb{C}^{\infty}$. □
The above result can also be obtained by transforming the results of Kristalny [9] to the sampled-data setting. Condition 1 in Proposition 1 says that if an instability of $\bar{G}_e$ does not "belong" to the space $\text{Im} \bar{H}$ then we cannot cancel them by choice of $\hat{S}$. Existence of a factorization of the form (9) in Condition 2 roughly speaking says that instabilities of $\bar{G}_e$ must be contained in $\bar{G}_v$. These two conditions are sufficient and necessary to obtain a stable sampler $\hat{S}$ (i.e. $\hat{S} \in L^\infty$) such that $\bar{G}_e \in L^\infty$. To obtain $\ell$-causal and stable sampler $\hat{S}$ (i.e. $\hat{S} \in z^iH^\infty$) we need an extra condition that there exists a $\bar{V} \in L^\infty$ such that $\bar{M}_h := \bar{H}_i^\infty \bar{M}_v - \bar{V} \bar{M}_y$ is in $z^iH^\infty$ (Condition 3 in Proposition 1). There may exist several such $\bar{V}$’s, so let us define the subspace $\mathcal{W} := \{\bar{V} \in L^\infty : \bar{M}_h \in z^iH^\infty\}$. Now, we show that for any two $\bar{V}_1, \bar{V}_2 \in \mathcal{W} \subseteq L^\infty$, $\text{proj}_{\mathcal{W} \cap z^iH^\infty}(\bar{V}_1 - \bar{V}_2) = 0$. This is used later in Proposition 2 to obtain a parameterization of all solutions of the stabilization problem in a single parameter. Note that $\bar{V}$ is a sampler, therefore, if it is in $L^\infty$ then it is in $L^2$ [13, proposition 5.3]. Hence, it makes sense to use the projection of a $\bar{V} \in \mathcal{W}$.

**Lemma 1** If $\bar{V}_1, \bar{V}_2 \in L^\infty$ are such that $\bar{H}_i^\infty \bar{M}_v - \bar{V}_1 \bar{M}_y \in z^iH^\infty$ (i = 1, 2) then,

$$\text{proj}_{z^iH^\infty}(\bar{V}_1 - \bar{V}_2) = 0.$$

where $\bar{M}_h, \bar{M}_v$ and $\bar{M}_y$ are defined in Proposition 1.

Lemma 1 is utilized in the following result. The proof is similar to the proof of [20, lemma 1].

**Proposition 2** If all the conditions of Proposition 1 are satisfied, then all samplers $\hat{S} \in z^iH^\infty$ such that $\bar{G}_e := \bar{G}_v - \bar{H} \bar{S} \bar{G}_y \in L^\infty$ can be parameterized in parameter $\hat{S}_\alpha \in z^iH^\infty$ as

$$\hat{S} = H_\alpha^{-1}(\hat{S}_\alpha \hat{S}_\nu - \hat{M}_h)$$

where $\hat{M}_h := H_\infty \hat{M}_v - \bar{V} \hat{M}_y$. In this case

$$\bar{G}_e = \bar{G}_v + H_1 \hat{M}_h \bar{G}_y - H_1 \hat{S}_\alpha \bar{S}_\nu$$

After solving the stabilization problem in Proposition 2, we can now concentrate on the optimization problem. For this, we need the following assumption:

**Assumption A_3**: $\bar{N}_y(e^{i\theta})\bar{N}_y(e^{i\theta})^* > 0$ for all $\theta \in [-\pi, \pi]$.

Assumption $\text{A}_3$ along with Assumption $\text{A}_1$ is essential to make $\bar{N}_y$ co-inner (i.e. $\bar{N}_y \bar{N}_y^\infty = I$) in (9). Now, we provide a solution to the Problem 2 in the following lemma:

**Proposition 3** Let assumptions $\text{A}_1$-$\text{A}_3$ be satisfied. If the stabilization problem has a solution, then

1. there exists a coprime factorization over $H^\infty$ of $\bar{G} := [\bar{G}_v \bar{G}_y]^T$ of the form (9) with co-inner $\bar{N}_y$.
2. $\bar{G}_e \in L^2$ iff $\Pi_{\bar{H}} \bar{G}_e \in L^2$ where $\Pi_{\bar{H}} := 1 - \bar{H} \bar{H}^\infty$. In that case, there is a unique sampler that solves

$$\hat{S}_{\text{opt}} := \arg\inf_{\hat{S} \in z^iH^\infty} \|\bar{G}_v - \bar{H} \hat{S} \bar{G}_y\|_{L^2} = H_\alpha^{-1}(\hat{S}_{\alpha, \text{opt}} \hat{M}_y - \hat{M}_h)$$

where

$$\hat{S}_{\alpha, \text{opt}} = \text{proj}_{z^iH^\infty}(\bar{H}_i^\infty \hat{N}_y \bar{N}_y^\infty - \bar{V}).$$

Moreover,

$$\|\bar{G}_{e, \text{opt}}\|_{L^2}^2 := \|\bar{G}_v - \bar{H} \hat{S}_{\text{opt}} \bar{G}_y\|_{L^2}^2$$

$$= \|\bar{G}_v + H_1 \hat{M}_h \bar{G}_y\|_{L^2}^2 - \|\hat{S}_{\alpha, \text{opt}}\|_{L^2}^2$$

Note that $\bar{H}$ is a hold therefore we can never take $\bar{H} = I$, in other words we will never have $\Pi_{\bar{H}} = 0$.

Our aim in the rest of this paper is to apply the results of Section 6 to a sampled-data setup where the signal generator and hold are given in state-space.

---

1 This condition is with the constraint that $\bar{G}_v$ is causal.
7 State-space of linear $h$-time shift invariant systems

In this section, we define state-space representation of linear $h$-time shift invariant systems and list some of the properties that are useful in obtaining the optimal relaxed causal sampler. State-space representation of LCTI systems is well-known. In order to understand the meaning of state-space for linear $h$-time shift invariant systems, we need concept of state-space with two point boundary condition (STPBC). All the proofs of the results in this section are given in Appendix 11.2.

7.1 State-space with two point boundary condition (STPBC)

Mirkin introduced STPBC for lifted systems in [17] using the earlier work of Krener [8], and Golberg and Kaashoek [3]. In this section, we briefly describe STPBC and summarize some of its properties. For details see [17, 21, 23].

A system $G$ mapping $u \in L^2[0,h)$ to $y \in L^2[0,h)$ is a system with two point boundary condition if it defined by the linear differential equations as

$$\begin{align*}
    \dot{x}(\tau) &= Ax(\tau) + B(\tau)u(\tau) \\
y(\tau) &= C(\tau)x(\tau) + Du(\tau)
\end{align*}$$

with boundary condition

$$\Omega x(0) + \Upsilon x(h^-) = 0$$

where $\tau \in [0,h)$. Here for integers $k, m$ and $n$, $A, \Omega, \Upsilon \in \mathbb{C}^{n \times n}$, $D \in \mathbb{C}^{k \times m}$, $B \in \mathbb{C}^2([0,h], \mathbb{C}^{n \times m})$ and $C \in \mathbb{C}^2([0,h], \mathbb{C}^{k \times n})$. The above representation of systems is known as state-space with two point boundary condition (STPBC). The system $G$ given by (15) is represented by the following notation in this paper

$$y = \begin{bmatrix} AB \\ CD \end{bmatrix} \begin{bmatrix} \Omega & \Upsilon \end{bmatrix} u \quad \text{(16)}$$

The usefulness of the STPBC representation is already established in the [17, 16, 21, 12].

These linear differential equations are defined well-posed if the output $y$ is uniquely determined by the input $u$ [8, 3]. It is shown in [8, 3] that well-posedness is equivalent to invertibility of the matrix

$$\Xi_G := \Omega + T e^{Ah}.$$ 

If (15) is well posed then the output $y$ is given by

$$y(\tau) = Du(\tau) + \int_0^h K_G(\tau, \sigma)u(\sigma)d\sigma \quad \text{(17)}$$

where

$$K_G(\tau, \sigma) = \begin{cases} C(\sigma)e^{A\tau} \Xi_G^{-1} \Omega e^{-A\sigma}B(\sigma) & \text{if } 0 \leq \sigma < \tau \leq h \\ -C(\tau)e^{A\tau} \Xi_G^{-1} \Upsilon e^{A(h-\sigma)}B(\sigma) & \text{if } 0 \leq \tau < \sigma \leq h \end{cases} \quad \text{(18)}$$

Alternatively, the output $y(\tau)$ can be written as

$$y(\tau) = Du(\tau) - C(\tau)\int_0^\tau e^{A(\tau-\sigma)}B(\sigma)u(\sigma) \, d\sigma + C(\tau)e^{A\tau} \Xi_G^{-1} \Omega \int_0^h e^{-A\sigma}B(\sigma)u(\sigma) \, d\sigma \quad \text{(19a)}$$

$$\quad = Du(\tau) + C(\tau)\int_\tau^h e^{A(\tau-\sigma)}B(\sigma)u(\sigma) \, d\sigma - C(\tau)e^{A\tau} \Xi_G^{-1} \Upsilon \int_0^h e^{A(h-\sigma)}B(\sigma)u(\sigma) \, d\sigma. \quad \text{(19b)}$$

The $y(\tau)$ given in (19) is sometimes more useful than $y(\tau)$ given in (17).

We say that two STPBCs are equivalent if for the same input, the output of both systems are equal in $L^2$ sense. We denote the equivalence with $\equiv$ symbol.

Now we list few frequently used operations related to STPBC without proof (see [17] for detail).

Lemma 2 Let the systems $G_1$ and $G_2$ be given by

$$G_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} \Omega_i & \Upsilon_i \end{bmatrix} \quad \text{(20)}$$

where $i = \{1, 2\}$. Also, let $G$ be given by STPBC (16). If $G$ and $G_i$ are well-posed, then
1. Similarity transformation

\[
\begin{bmatrix}
TAT^{-1} & TB \\
CT^{-1} & D
\end{bmatrix}
\begin{bmatrix}
SOT^{-1} & SYT^{-1}
\end{bmatrix} \equiv G
\]

Here \(S\) and \(T\) are invertible matrices.

2. Parallel interconnection

\[
G_1 + G_2 = \begin{bmatrix}
A_1 & B_1 \\
0 & A_2
\end{bmatrix}
\begin{bmatrix}
B_1 & B_2 \\
C_1 & C_2
\end{bmatrix}
\begin{bmatrix}
D_1 + D_2 \\
0 & D_2
\end{bmatrix}
\begin{bmatrix}
\Omega_1 & 0 \\
0 & \Omega_2
\end{bmatrix}
\begin{bmatrix}
\mathbb{I} \\
0
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2
\end{bmatrix}
\]

3. Series interconnection

\[
G_1 G_2 = \begin{bmatrix}
A_1 & B_1 C_2 \\
0 & A_2
\end{bmatrix}
\begin{bmatrix}
B_1 & B_2 \\
C_1 & C_2
\end{bmatrix}
\begin{bmatrix}
D_1 + D_2 \\
0 & D_2
\end{bmatrix}
\begin{bmatrix}
\Omega_1 & 0 \\
0 & \Omega_2
\end{bmatrix}
\begin{bmatrix}
\mathbb{I} \\
0
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2
\end{bmatrix}
\]

All of these systems are well-posed as well. □

Note that if we left-multiply the boundary conditions with an invertible matrix \(S\) then we get an equivalent system (use \(T = I\) in the similarity transformation equation above). For detail see [23, corollary 5.3.4].

7.2 Systems in STPBC

Now, we present STPBCs of different linear \(h\)-time shift invariant systems including LCTI systems, holds and samplers. Just like the state-space does not represent all LCTI systems, STPBCs do not represent all linear \(h\)-time shift invariant systems. However, it represents a fairly large class of systems including rational LCTI systems. A generic linear \(h\)-time shift invariant system \(y = G u\) mapping \(L^2(\mathbb{R})\) to \(L^2(\mathbb{R})\) that can be represented by STPBC is given by

\[
\ddot{y}(z) = \tilde{G}(z)\ddot{u}(z) : \quad \ddot{y}(z) = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
\Omega(z) \\
\mathbb{T}(z)
\end{bmatrix}
\ddot{u}(z),
\]

in lifted \(z\)-domain. Here \(A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^2([0,h], \mathbb{C}^{n \times m}), C \in \mathbb{C}^2([0,h], \mathbb{C}^{k \times n}), D \in \mathbb{C}^{k \times m}\) for integers \(k, m\) and \(n\), and \(\Omega(z)\) and \(\mathbb{T}(z)\) are square discrete transfer matrices. \(D\) is known as direct feed-through term of the STPBC representation of system \(\tilde{G}\) (or in short, direct feed-through term of the system \(\tilde{G}\)). For example, rational LCTI system given in state space have \(\Omega(z) = zI\) and \(\mathbb{T}(z) = -I\) [16].

With some modification, STPBCs can be used to represent holds and samplers also. Assume \(\tilde{G}(z)\) with \(D = 0\) in (21). Then we can use

\[
\begin{align*}
\ddot{H}(z) &= \tilde{G}(z)\mathcal{J}_r \\
\dot{S}(z) &= \mathcal{J}_r\tilde{G}(z)
\end{align*}
\]

(22)

(23)

to represent hold and sampler in lifted domain. Here the impulse operator \(\mathcal{J}_r\) is defined as

\[
\mathcal{J}_r \eta := \delta(\tau - r)\eta \quad \eta \in \mathbb{C}^n, \quad \tau, r \in (0, h)
\]

(24)

and sampling operator \(\mathcal{J}_r^*\) is defined as

\[
\mathcal{J}_r^* g := g(r) \quad g \in \mathbb{C}^2([0,h]).
\]

(25)

As long as \(g\) is continuous we can treat \(\mathcal{J}_r^*\) as the adjoint of \(\mathcal{J}_r\) and vice-versa [17]. For a detailed discussion on the operators \(\mathcal{J}_r^*\) see [17].
7.3 Conjugate of the system given in STPBC

The conjugate of the system $\tilde{G}(z)$ is defined as $\tilde{G}^\sim(z) := [G(z^{-1})]^*$ [16]. It can be shown that conjugate $\tilde{G}^\sim(z)$ of the system $\tilde{G}(z)$ given in (21) has STPBC (see [16] for details)

$$\tilde{G}^\sim(z) = \begin{bmatrix} -A^* & C^* \\ -B^* & D^* \end{bmatrix} \begin{bmatrix} T_{a_d}(z) & \Omega_d^*(z) \end{bmatrix}$$

where $\Omega_d(z)$ and $T_d(z)$ are any square discrete transfer matrices satisfying

$$\Omega(z)T_d(z) = \Upsilon(z)\Omega_d(z)$$

and such that $\begin{bmatrix} \Omega_d(z) \\ T_d(z) \end{bmatrix}$ has full normal rank.

Similarly using [17, lemma 2], we can show that the conjugate $\hat{H}^\sim$ of hold $\hat{H}$ in (22) is given by $\hat{H}^\sim = J^*\tilde{G}^\sim$, and the conjugate $\hat{S}^\sim$ of sampler $\hat{S}$ in (23) is given by $\hat{S}^\sim = \tilde{G}^\sim J_r$. Note that in STPBC of sampler and hold, the direct feed-through term $D$ of the system $\tilde{G}$ is assumed zero.

7.4 Stability and causality of systems given in STPBC

In this section, we briefly describe the stability and causality condition of a system given in STPBC. Note that if a system is stable and causal then the system is in $H^\infty$. Most of the causual systems discussed later in this chapter have boundary conditions $\Omega(z) = zI$ and $T(z) = -\Upsilon \in \mathbb{C}^{n \times n}$. Therefore, we restrict ourself to these boundary conditions.

**Lemma 3** Let $\tilde{G}$, $\hat{H}$ and $\hat{S}$ be a system with STPBC given by (21), (22) and (23) respectively. Now, If $\Omega(z) = zI$ and $T(z) = -\Upsilon \in \mathbb{C}^{n \times n}$ then $\tilde{G}$, $\hat{H}$ and $\hat{S}$ are in $H^\infty$ if $Te^{Ah}$ is Schur (i.e. having eigenvalues in $\mathbb{D}$). □

7.5 $H^2$ norm of systems given in STPBC

This section is devoted to the $H^2$ norm of systems that are represented as STPBC. It is well known that if the direct feed through term of an analog system given in state-space is not zero, then that system does not have a finite $H^2$ norm. The same can be said about systems represented by STPBC (see also [12, remark 23.2]).

**Lemma 4** Let $\tilde{G}$ is a system with STPBC given by (21). Now, if $D \neq 0$ then $\tilde{G} \notin L^2$. □

Most of the systems considered in this paper are in $H^\infty$ and for such systems we have the following simple result.

**Lemma 5** Let $\tilde{G}$ be a system with STPBC given by (21). Also, let $D$ be as in (21) and assume $D = 0$. Then $\tilde{G} \in H^\infty$ implies $\tilde{G} \in H^2$. □

**Remark 2** Lemma 5 holds even if $H^\infty$ and $H^2$ are replaced by $L^\infty$ and $L^2$ respectively. □

Similar to Section 7.4, in this section we calculate the $H^2$ norm of systems that have $\Omega(z) = zI$ and $T(z) = -\Upsilon \in \mathbb{C}^{n \times n}$. These systems are such that their $H^2$ norm is equal to the $H^2$ norm of a discrete system.

**Lemma 6** Let $\tilde{G}$ be a causal system with STPBC given by (21). Also, let $A$, $B$, $C$, $D$, $\Omega(z)$ and $T(z)$ be as in (21). Now, there exists matrices $\hat{B}$ and $\hat{C}$ that satisfy

$$\hat{B}\hat{B}^* := \int_0^h e^{A(h-\sigma)}B(\sigma)B(\sigma)^*e^{A^*(h-\sigma)}\, d\sigma, \quad \hat{C}^*\hat{C} := \int_0^h e^{A^*(\tau)}C(\tau)C(\tau)^*e^{A\tau}\, d\tau. \quad (27a)$$

**Assume** $D = 0$, $\Omega(z) = zI$ and $T(z) = -\Upsilon \in \mathbb{C}^{n \times n}$. If $\tilde{G} \in H^2$ then the squared $H^2$ norm of $\tilde{G}$ equals

$$||\tilde{G}||^2_{H^2} = \frac{1}{h}||\hat{D}||^2_{HS} + ||\hat{Y}||^2_{H^2}. \quad (28)$$
where $\tilde{D} : L^2[0, h) \rightarrow L^2[0, h)$ is given by
\[
\tilde{y} = \tilde{D}\tilde{u} : \quad \tilde{y}(\tau) = \int_{0}^{h} C(\tau) e^{A(\tau - \sigma)} B(\sigma) \mathbb{1}(\tau - \sigma) \tilde{u}(\sigma) d\sigma
\] (29)
and $\tilde{Y}$ is a discrete system with
\[
\tilde{Y}(z) = \tilde{C}(zI - \tilde{T}e^{Ah})^{-1}\tilde{Y}
\]
The squared Hilbert-Schmidt norm of $\tilde{D}$ is given by
\[
\|\tilde{D}\|_{HS}^2 = \text{tr} \int_{0}^{h} \int_{0}^{h} C(\tau) e^{A(\tau - \sigma)} B(\sigma) B(\sigma)^* e^{A^*(\tau - \sigma)} C^*(\tau) \mathbb{1}(\tau - \sigma) d\sigma d\tau
\] (30)

In a similar way, we can state the following result about $H^2$ norm of holds and samplers.

**Lemma 7** Let $A, B, C, D, \Omega(z)$ and $T(z)$ be as in Lemma 3. Let matrices $\tilde{B}$ and $\tilde{C}$ be such that they satisfy (27a) and (27b) respectively. Assume $D = 0$, $\Omega(z) = zI$ and $T(z) = -T \in \mathbb{C}^{n \times n}$. Now, for $r \in (0, h)$

1. if a causal hold $\tilde{H}$ is given by (22) then
\[
\|\tilde{H}\|_{H^2} = \|\tilde{Y}_H\|_{H^2}
\]
where $\tilde{Y}_H$ is a discrete system with
\[
\tilde{Y}_H(z) = \tilde{C}(zI - \tilde{T}e^{Ah})^{-1}\tilde{B}(r)
\]
2. if a causal sampler $\tilde{S}$ is given by (23) then
\[
\|\tilde{S}\|_{H^2} = \|\tilde{Y}_S\|_{H^2}
\]
where $\tilde{Y}_S$ is a discrete system with
\[
\tilde{Y}_S(z) = C(r)(zI - \tilde{T}e^{Ah})^{-1}\tilde{Y}
\]

**Remark 3** The adjoint of an anti-causal system is causal, therefore $L^2$ norm of the anti-causal system can be calculated by using lemmas 6 and 7.

### 7.6 Computations

Integrals given in (27) and (30) seems very tedious to evaluate. However if $B$ and $C$ are constant then these integrals can be calculated using matrix exponentials [10, 2, 17].

**Lemma 8** If $A$ is a square constant matrix, and $B$ and $C$ are constant matrices of appropriate dimensions then
\[
\int_{0}^{h} e^{As}BB^*e^{As} ds = \Gamma_{33}(A, B)\Gamma_{23}(A, B)
\]
\[
\int_{0}^{h} \int_{0}^{t} e^{As}BB^*e^{As} dsdt = \Gamma_{33}(A, B)\Gamma_{13}(A, B)
\]
\[
\int_{0}^{h} e^{As}Ce^{As} ds = \Lambda_{22}(A, C)\Lambda_{12}(A, C)
\]
where
\[
\Gamma(A, B) = \begin{bmatrix}
\Gamma_{11}(A, B) & \Gamma_{12}(A, B) & \Gamma_{13}(A, B) \\
0 & \Gamma_{22}(A, B) & \Gamma_{23}(A, B) \\
0 & 0 & \Gamma_{33}(A, B)
\end{bmatrix} = \exp\begin{bmatrix}
-A & I & 0 \\
0 & -A & BB^* \\
0 & 0 & A^*
\end{bmatrix} h
\]
and
\[
\Lambda(A, C) = \begin{bmatrix}
\Lambda_{11}(A, C) & \Lambda_{12}(A, C) \\
0 & \Lambda_{22}(A, C)
\end{bmatrix} = \exp\begin{bmatrix}
-A^* & C^*C \\
0 & A
\end{bmatrix} h
\]

**Proof** The proof is given in [2].
8 STPBC solution

In this section we apply the results of Section 6 to a sampled-data setup where the signal generator and hold are given in state-space with two-point boundary condition (STPBC). All the proofs of the results in this section are given in Appendix 11.3.

We assume that the causal signal generator $\mathcal{G}$ in Laplace domain is given by,

$$G(s) = \begin{bmatrix} G_v(s) \\ G_y(s) \end{bmatrix} = D + C(sI - A)^{-1}B$$

where $C := \begin{bmatrix} C_v \\ C_y \end{bmatrix}$ and $D := \begin{bmatrix} 0 \\ D_y \end{bmatrix}$. The feed-through term $G_v(\infty)$ is taken zero so that we have bounded $\|G_0\|_{L^2}$ (see Lemma 4). This is an LCTI system therefore the STPBC of $\mathcal{G}$ in lifted $z$-domain is given by (see [16]):

$$\tilde{G}(z) = \begin{bmatrix} \tilde{G}_v(z) \\ \tilde{G}_y(z) \end{bmatrix} = \begin{bmatrix} A & B \\ C_v & 0 \\ C_y & D_y \end{bmatrix}[zI - I].$$

Without loss of generality, we assume that $A = \text{diag}\{A_s, A_u\}$, where $A_s$ has all its eigenvalues in $\mathbb{C}^- := \{z \in \mathbb{C} : \text{real}(z) < 0\}$ and $A_u$ has all its eigenvalues in $\mathbb{C}\backslash\mathbb{C}^-$. Also let $C_v := [C_{vs} \ C_{vu}]$ be the partition of $C_v$ according to $A_s$ and $A_u$. Therefore

$$\tilde{G}(z) = \begin{bmatrix} A_s & 0 \\ 0 & A_u \\ C_{vs} & C_{vu} \\ C_y & 0 \\ D_y \end{bmatrix}[zI - I].$$

Note that systems $\mathcal{G}_v$ and $\mathcal{G}_y$ are causal (but not necessarily stable) by assumption. However, to obtain an optimal sampler described in Proposition 3, in addition to causality, we need that $\mathcal{G}_y$ must be rational and proper (Assumption $A_4$). Since $\mathcal{G}_y$ is represented in state-space, it is rational and proper. Also, we need that there exists a left coprime-factorization of $\tilde{G}$ of the form (9). To this end, we need the following assumption.

Assumption $A_4$: $(C_y, A)$ is observable and $(A, B)$ is controllable.

Later it will be explained in Section 8.1 that $A_4$ allows the existence of a coprime factorization, and assumptions $A_3$ and $A_4$ allow the existence of a coprime factorization $\tilde{G}_y = \tilde{M}_y^{-1} \tilde{N}_y$ with $\tilde{N}_y$ co-inner.

Also, we consider hold $\hat{H}$ with STPBC given by

$$\hat{H}(z) := \begin{bmatrix} A_H & B_H \\ C_H & 0 \end{bmatrix}[zI - E].$$

where impulse operator $J_{0^+}$ defined in (24) is needed to perform the discrete to analog domain conversion. The holds given by STPBC (34) can represent a large class of stable holds with infinite or finite impulse response. For example the ideal zero order hold $H_a$ (i.e. with hold function $\mathbb{1}_{[0, h]}(t)$) can be obtained by setting $I = C_H = B_H$ and $0 = A_H = E$. We also assume the following about $\hat{H}$:

Assumption $A_5$: $E e^{A_h h}$ is a Schur matrix,

Assumption $A_6$: $B_H$ has full column rank.

Assumption $A_7$: $(C_H, A_H)$ is observable.

$E e^{A_h h}$ is a Schur matrix is just a restatement of the fact that $\hat{H} \in H^\infty$ (see Lemma 3). Assumption $A_5$ allow us to obtain a right coprime factorization of $\hat{H}$ and assumptions $A_5$-$A_7$ allow us to obtain an inner-outer factorization of $H$. This is explained in Section 8.3.

From now onwards we drop the the subscript $z$ from the lifted transfer functions unless necessary. We also do it for the signal.
8.1 Left coprime-factorization of $\tilde{G}$

The following lemma describes the left coprime-factorization over $H^\infty$ of $\tilde{G}$ which is required in Proposition 3.

**Lemma 9** Let \[ \begin{bmatrix} \tilde{G}_v \\ \tilde{G}_y \end{bmatrix} \] have STPBC given in (32). If Assumption $A_4$ is satisfied then there exists an $L$ such that $A + LC_y$ is Hurwitz. In that case

\[
\begin{bmatrix} \tilde{G}_v \\ \tilde{G}_y \end{bmatrix} = \begin{bmatrix} I & M_v \\ 0 & M_y \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_v \\ \tilde{N}_y \end{bmatrix}
\]  

(35)

for

\[
\begin{bmatrix} M_v & N_v \\ M_y & N_y \end{bmatrix} := \begin{bmatrix} A + LC_y & L & B + LD_y \\ C_y \\ 0 \\ Z_y & Z_y D_y \end{bmatrix} [zI - I]
\]  

(36)

where $Z_y$ is any invertible complex matrix. In this case, $\tilde{M}_y$ and $\tilde{N}_y$ are left coprime, and $\tilde{M}_v, \tilde{N}_v$ belong to $H^\infty$. □

We also need that $\tilde{N}_y$ to be co-inner in the Proposition 3. Here, Assumption $A_3$ helps. We start with the following standard result to check Assumption $A_3$ in the state space.

**Lemma 10** Let $\tilde{N}_y$ be as in (36). If $(C_y, A)$ is observable (see $A_4$) then Assumption $A_3$ (i.e. $\tilde{N}_y(e^{i\theta})\tilde{N}_y^*(e^{i\theta}) > 0 \forall \theta \in [-\pi, \pi]$) is satisfied iff $D_y$ has full row rank and

\[
\begin{bmatrix} A - j\omega I & B \\ C_y & D_y \end{bmatrix}
\]

has full row rank for all $\omega \in \mathbb{R}$. □

The following result explains how to do the left coprime factorization $\tilde{G}_y = \tilde{M}_y^{-1}\tilde{N}_y$ with $\tilde{N}_y$ co-inner.

**Lemma 11** If assumptions $A_3$ and $A_4$ are satisfied then by Lemma 9 there exists a coprime factorization of $\tilde{G}$ given in (32) of the form (35), $R := D_y D_y^*$ is invertible and there exists a unique stabilizing solution $X$ (i.e. such that matrix $A + (-Y C_y + B D_y^*) R^{-1} C_y$ is Hurwitz) of the Riccati equation

\[ AX + XA^* - (X C_y^* + B D_y^*) R^{-1} (C_y X + D_y B^*) + BB^* = 0. \]

If we choose

\[
Z_y = R^{-\frac{1}{2}}  \\
L = -(X C_y^* + B D_y^*) R^{-1}
\]

then $\tilde{N}_y$, defined in (36) is co-inner. □

The following lemma is useful later in obtaining the optimal sampler.

**Lemma 12** Let $L$, $Z_y$ and $X$ be as in Lemma 11, and $\tilde{N}_v$ and $\tilde{N}_y$ as in Lemma 9. Now,

\[
\tilde{N}_v \tilde{N}_y^* = \begin{bmatrix} -(A + LC_y) & (Z_y C_y)^* \\ -C_v X & 0 \end{bmatrix}
\]

□
8.2 Simplification of Assumption $\mathcal{A}_2$

It is desirable to have a simple criterion which tells us that Assumption $\mathcal{A}_2$ (i.e., assumption of existence of an inner-outer factorization of $\tilde{H} = \tilde{H}_1\tilde{H}_o$ with inner $\tilde{H}_1 \in H^\infty$, and bistable and bicausal $\tilde{H}_o \in H^\infty$) is satisfied or not. To this end, using (19) we write the hold $\tilde{H}(z)$, defined in (34), as

$$\tilde{H}(z) = \begin{bmatrix} \bar{A}_H & B_H \\ C_H & 0 \end{bmatrix} \begin{bmatrix} zI - E \end{bmatrix} \mathcal{J}_{0^+}$$

$$= C_H e^{A_H \tau} \left( I + (zI - Ee^{A_H h})^{-1} Ee^{A_H h} \right) B_H$$

$$= C_H e^{A_H \tau} \tilde{H}_s(z)$$

where $\tilde{H}_s$ is a discrete system which is rational in $z$ and it is given by

$$\tilde{H}_s(z) := \begin{pmatrix} \frac{Ee^{A_H h}B_H}{1} & Ee^{A_H h}B_H \\ I & B_H \end{pmatrix}$$

Hence,

$$\tilde{H}^\sim(z)\tilde{H}(z) = \tilde{H}_s(z)^\sim\bar{C}_H \bar{C}_H \tilde{H}_s(z)$$

where $\bar{C}_H$ is a matrix which satisfies

$$\bar{C}_H \bar{C}_H = \int_0^h e^{A_H \tau} C_H e^{A_H h} d\tau.$$  \hfill (39)

The following proposition explains the relationship of Assumption $\mathcal{A}_2$ with other criteria.

**Proposition 4** Let $\bar{C}_H$ and $\tilde{H}_s(z)$ as in (39) and (38) respectively. If the hold $\tilde{H} \in H^\infty$ given by STPBC (34) satisfies Assumption $\mathcal{A}_5$ (i.e., $Ee^{A_H h}$ is Schur) then the following are equivalent:

1. The inner outer factorization of hold $\tilde{H} = \tilde{H}_1\tilde{H}_o$ exists with inner $\tilde{H}_1 \in H^\infty$, and bistable and bicausal $\tilde{H}_o \in H^\infty$ (Assumption $\mathcal{A}_2$).
2. The spectral factorization of $\tilde{H}^\sim \tilde{H}$ exists i.e. there exists a bistable and bicausal spectral factor $\tilde{W}$ such that $\tilde{H}^\sim \tilde{H} = \tilde{W}^\sim \tilde{W}$.
3. $\tilde{H}^\sim(e^{i\theta})\tilde{H}(e^{i\theta}) > 0$ i.e. the matrix

$$\begin{bmatrix} Ee^{A_H h} - e^{i\theta} I & B_H \\ C_H & 0 \end{bmatrix}$$

has full column rank for every $\theta \in [-\pi, \pi]$.
4. The discrete algebraic Riccati equation

$$Q_0 = e^{A_H h}E^*(Q_0 - Q_0B_H(B_H^*Q_0B_H)^{-1}B_H^*Q_0)e^{A_H h} + \bar{C}_H^* \bar{C}_H$$

has a unique solution $Q_0$ for which $(E - B_H(B_H^*Q_0B_H)^{-1}B_H^*Q_0E)e^{A_H h}$ is Schur-stable.

$\square$

Thus, using Proposition 4, we can easily check that Assumption $\mathcal{A}_2$ is satisfied or not. We also use Proposition 4 later in obtaining an inner-outer factorization of the hold.

Now, we show that Assumption $\mathcal{A}_2$ is satisfied if assumptions $\mathcal{A}_5$ - $\mathcal{A}_7$ are satisfied.

**Lemma 13** Let STPBC of hold $\tilde{H}$ be given by (34). If assumptions $\mathcal{A}_5$ - $\mathcal{A}_7$ are satisfied then there exists a factorization of $\tilde{H} = \tilde{H}_1\tilde{H}_o$ with inner $\tilde{H}_1 \in H^\infty$, and bistable and bicausal $\tilde{H}_o \in H^\infty$ (i.e. Assumption $\mathcal{A}_2$ is satisfied). $\square$
8.3 Inner-outer factorization of hold

There are many ways of obtaining an inner-outer factorization for \( \hat{H} \) given in (34). We adopted the method used by [21] i.e we first obtain a right coprime factorization (RCF) over \( H^\infty \) of \( H = \bar{N}_I \bar{M}_I^{-1} \) and then we make \( \bar{N}_I \) inner (i.e. \( \bar{N}_I \bar{N}_I = I \)). \( \bar{N}_I \) and \( \bar{M}_I \) are said to be coprime in \( H^\infty \) if \( \bar{N}_I \) and \( \bar{M}_I \) are in \( H^\infty \) and there exist Bezout factors \( \bar{X}_I \in H^\infty \) and \( \bar{Y}_I \in H^\infty \) such that

\[
X_I \bar{M}_I + Y_I \bar{N}_I = I.
\]

We start with a right coprime factorization (RCF) of \( \hat{H} \).

**Lemma 14** Consider the hold \( \hat{H} \) given by (34). Suppose that Assumption \( \mathcal{A}_5 \) is satisfied and assume that there exists a non-zero matrix \( F \) such that \( (E + B_H F)e^{A_I h} \) is Schur. Now, the Hold \( \hat{H} = \bar{N}_I (\bar{M}_I)^{-1} \), where \( \bar{N}_I \in H^\infty \) and \( \bar{M}_I \in H^\infty \) are right coprime and given by

\[
\begin{bmatrix}
\hat{M}_I(z) \\
\hat{N}_I(z)
\end{bmatrix} = \begin{bmatrix}
A_I & B_H \mathcal{J}_0^+ \\
\frac{1}{2} J_N F & C_H \\
0 & I
\end{bmatrix} \begin{bmatrix} I \\ zI -(E + B_H F) \end{bmatrix} \tag{40}
\]

Also, \( \bar{M}_I^{-1} \in H^\infty \). \( \square \)

Note that if \( \mathcal{A}_5 \) is satisfied, then \( F = 0 \) makes \( (E + B_H F)e^{A_I h} \) Schur in the above lemma. However, there may be many such \( F \) other than zero. Now, we concentrate on exploiting this \( F \) to make \( \bar{N}_I \) (given in Lemma 14) inner. This is because if \( \bar{N}_I \) is inner then the inner factor \( \bar{H}_I \) of \( \hat{H} = \bar{N}_I \bar{M}_I^{-1} \) and the bicausal and bistable factor \( \bar{H}_o \) of \( \hat{H} \) is \( \bar{M}_I^{-1} \). Note that \( \bar{M}_I^{-1} \) is stable and causal if the hold \( \hat{H} \) is stable and causal.

Since \( \hat{H} \) is assumed to be in \( H^\infty \), \( F = 0 \) renders a trivial RCF i.e \( \bar{N}_I = \hat{H} \) and \( \bar{M}_I = I \). However, in general \( \hat{H} \) is not inner. Therefore, we need an non-trivial \( F \) to make \( \bar{N}_I \) inner.

**Lemma 15** (Inner-outer factorization of the Hold) Consider the Hold \( \hat{H} \) given by (34) and suppose that assumptions \( \mathcal{A}_5 \), \( \mathcal{A}_7 \) are satisfied. Then, there exists a unique stabilizing solution \( Q_0 > 0 \) of the Riccati equation

\[
Q_0 = e^{A_I h} E^* (Q_0 - Q_0 B_H (B_H^* Q_0 B_H)^{-1} B_H^* Q_0) E e^{A_I h} + C_H^* C_H
\]

where \( C_H \) is a matrix which satisfies (39) and \( B_H^* Q_0 B_H \) is invertible. If we define

\[
\begin{align*}
Z & := (B_H^* Q_0 B_H)^{-\frac{1}{2}} \\
F & := -(B_H^* Q_0 B_H)^{-1} B_H^* Q_0 E \\
\bar{H}_I & := \bar{N}_I Z \\
\bar{H}_o^{-1} & := \bar{M}_I Z
\end{align*}
\]

where \( \bar{M}_I \) and \( \bar{N}_I \) are given by (40), then \( \bar{H}_I \bar{H}_o \) forms an inner-outer factorization of \( \hat{H} \) with inner \( \bar{H}_I \in H^\infty \), and bistable and bicausal \( \bar{H}_o \in H^\infty \). \( \square \)

8.4 The condition \( (I - \bar{H}_I \bar{H}_o^{-1}) \bar{G}_v \in L^\infty \)

The first thing we need to check is the condition \( (I - \bar{H}_I \bar{H}_o^{-1}) \bar{G}_v \in L^\infty \) (see Proposition 1) for the existence of a solution of Problem 2. Note that we have to check \( (I - \bar{H}_I \bar{H}_o^{-1}) \bar{G}_v \in L^\infty \) with the constraint that \( \bar{G}_v \) is causal. A state space formulation of the condition can be done by using the STPBC of \( \bar{H}_I \) (the inner factor of hold \( \hat{H} \)). However, the construction of \( \bar{H}_I \) requires a Riccati equation (see Section 8.3). The main aim of this section is to check the condition \( (I - \bar{H}_I \bar{H}_o^{-1}) \bar{G}_v \in L^\infty \) with a computationally efficient method which does not require Riccati equations.

We need the following assumption in simplification of the condition \( (I - \bar{H}_I \bar{H}_o^{-1}) \bar{G}_v \in L^\infty \). We also need it later in Section 8.6 for obtaining \( \bar{V} \) needed in Proposition 1.

**Assumption \( \mathcal{A}_8 \)**: No pole of \( \bar{G}_v \) in the region \( |z| \geq 1 \) is a zero of the discrete system \( \bar{H}_cs : = \bar{C}_H \bar{H}_s \) (\( \bar{C}_H \) and \( \bar{H}_s \) are defined in (39) and (38) respectively).

Note that \( (I - \bar{H}_I \bar{H}_o^{-1}) \) is the orthogonal projection onto the space \( (\text{Im} H)^\perp \). This projection can be calculated in many ways (not necessarily by constructing \( I - \bar{H}_I \bar{H}_o^{-1} \)). We used this idea in the result below, however we directly present the final result here due to lengthly proof. For a detailed discussion on this topic see [23, §5.4.4].
Theorem 1 Let assumptions $A_5$, $A_8$ be satisfied. If $(A, B)$ is controllable and $\tilde{G}_v$ is causal then $(I - \hat{H}, \hat{H}^-)\tilde{G}_v \in L^\infty$ iff
\[
C^*_v C_{vu} - P^*_u (C^*_H C_H)^+ P_u = 0
\]
and there exists an $X_t$ that satisfies the following linear equations
\[
\begin{align*}
\left((E e^{A_H h} - E e^{A_H h} B_H \tilde{D}^+ C_H) X_t - X_t e^{A_H h} + E e^{A_H h} B_H \tilde{D}^+ = 0 \right. \\
- \tilde{D}^+ C_H X_t + \tilde{D}^+ (C_H H)^+ P_u = 0
\end{align*}
\]  
(44a)
\[
(44b)
\]
Here $C_H$ is matrix which satisfies (39), $C^*_v C_{vu} = \int_0^h e^{A_H^* \tau} C^*_u C_{vu} e^{A_H \tau} d\tau$, $P_u = \int_0^h e^{A_H^* \tau} C^*_u C_{vu} e^{A_H \tau} d\tau$ and $D^\delta := [D^+ \ D^-]^T$ is an invertible matrix such that $D^\delta C_H B_H = [I \ 0]^T$. \(\Box\)

Detailed proof of the above theorem is given in [23, theorem 5.4.28]. It can be proved that if $X_t$ exists in the above theorem then it is unique. The matrix integrals $C_H H, C^*_v C_{vu}$ and $P_u$ can be calculated by methods given in [10].

Remark 4 If Assumption $A_7$ is satisfied then the condition given in (43) can be further simplified as explained below. Note that
\[
\int_0^h e^{A_H^* \tau} C^*_u C_{vu} e^{A_H \tau} d\tau = \begin{bmatrix} C^*_v C_{vu} & P^*_u \\ P_u & C_H^* C_H \end{bmatrix}
\]
Here $\dim A_H$ means number of rows of the square matrix $A_H$. This implies that the condition given in (43) is satisfied iff
\[
\rank \int_0^h e^{A_H^* \tau} C^*_u C_{vu} e^{A_H \tau} d\tau = \dim A_H.
\]
Now, $\rank \int_0^h e^{A_H^* \tau} C^*_u C_{vu} e^{A_H \tau} d\tau$ is equal to the rank of the observability matrix associated with pair $(C_{vu}, A_{vu})$ (see [32, theorem 3.3.3.8]). Therefore, the condition given in (43) can be verified just by showing that the rank of the observability matrix associated with pair $(C_{vu}, A_{vu})$ is equal to $\dim A_H$.

8.5 The condition $(I - \hat{H}, \hat{H}^-)\tilde{G}_v \in L^2$

We also need to check the condition $(I - \hat{H}, \hat{H}^-)\tilde{G}_v \in L^2$ for the existence of a solution of Problem 2 (see Proposition 3). To this end Lemma 5 and Remark 2 are very useful and applied in the following result.

Corollary 1 Let system $\tilde{G}$ and $\hat{H}$ are given by STPBC (33) and (34) respectively. Then $(I - \hat{H}, \hat{H}^-)\tilde{G}_v \in L^\infty$ implies $(I - \hat{H}, \hat{H}^-)\tilde{G}_v \in L^2$.

Proof Since the direct feed through term of $\tilde{G}_v$ is 0, the STPBC of $(I - \hat{H}, \hat{H}^-)\tilde{G}_v$ has zero direct feed-through term. Therefore, it follows from Lemma 5 and Remark 2 that if $(I - \hat{H}, \hat{H}^-)\tilde{G}_v \in L^\infty$ then $(I - \hat{H}, \hat{H}^-)\tilde{G}_v \in L^2$. 

8.6 Obtaining $\dot{V}$

Now, we concentrate on showing the existence of a $\dot{V} \in L^\infty$ such that $\dot{M}_h := \dot{H}_{\infty}^\sim \dot{M}_\nu - \dot{V} \dot{M}_\nu \in H^\infty$ (see Proposition 1). In this section, we show that

$$
\dot{V}(z) = J_0^+ \begin{bmatrix}
-\widetilde{A}_H & -A^*_H P(\tau) L Z_y^{-1} \\
-(B_H Z)^* & 0
\end{bmatrix} \begin{bmatrix}
0 \\
[z(E + B_H F)^* - I]
\end{bmatrix}
$$

(45)

is in $L^\infty$ and it is such that $\dot{M}_h \in H^\infty$, where $P(\tau)$ is the solution of Sylvester differential equation

$$
\dot{P}(\tau) + A^*_H P(\tau) + P(\tau) A + C^*_H C_v = 0, \quad \text{with boundary condition } (E + B_H F)^* P(0) = - P(h).
$$

(46)

First, we show that Sylvester differential equation (46) with boundary condition $(E + B_H F)^* P(0) = - P(h)$ has a solution. To this end, following Lemma is useful.

**Lemma 16** If Assumption $\mathcal{A}_8$ (given at page 14) is satisfied then no pole of $\hat{H}_{\infty}^\sim$ in the region $|z| \geq 1$ is a pole of $\hat{G}_v$.

We use the above Lemma to show that Sylvester differential equation (46) with boundary condition $(E + B_H F)^* P(0) = - P(h)$ has a solution.

**Lemma 17** If Assumption $\mathcal{A}_8$ is satisfied, a unique solution $P(\tau)$ of the differential Sylvester equation (46) with the boundary condition $(E + B_H F)^* P(0) = - P(h)$ exists and it is given by

$$
P(\tau) = A - A^*_H P_0 e^{-A \tau} - R_3(\tau),
$$

where

$$
R_3(\tau) := \int_0^\tau e^{-A^*_H \tau_1} C_H C_v e^{-A \tau_1} d\tau_1
$$

and $P_0$ is the unique solution of Sylvester equation

$$
P_0 = e^{A^*_H b} \begin{bmatrix}
-(E + B_H F)^* P_0 + R_3(h)
\end{bmatrix} e^{A h}.
$$

(47)

Now, we show that $\dot{V}$ defined in (45) is in $L^\infty$ and is such that $\dot{M}_h := \dot{H}_{\infty}^\sim \dot{M}_\nu - \dot{V} \dot{M}_\nu \in H^\infty$.

**Lemma 18** If assumptions $\mathcal{A}_3 - \mathcal{A}_8$ are satisfied then $\dot{V}$ defined in (45) exists and is in $L^\infty$. Also, $\dot{V}$ is such that $\dot{M}_h := \dot{H}_{\infty}^\sim \dot{M}_\nu - \dot{V} \dot{M}_\nu \in H^\infty$ and

$$
\dot{M}_h(z) = J_0^+ \begin{bmatrix}
A + LC_y \\
-(B_H Z)^* P_0
\end{bmatrix} \begin{bmatrix}
Z I - I
\end{bmatrix}
$$

(48)

8.7 Optimal relaxed causal sampler

In this section, we write a STPBC for optimal relaxed causal sampler described in Proposition 3. For the solution described in Proposition 3, we need $\dot{H}_{\infty}^\sim \dot{N}_v \dot{N}_v^\sim - \dot{V}$. Following corollary show how to obtain a compact STPBC for this.

**Corollary 2** Define $A_p := \begin{bmatrix}
-A^*_H & -C^*_H C_v X \\
0 & -A^*_L
\end{bmatrix}$, $B_p := \begin{bmatrix}
-P(\tau) L Z_y^{-1} \\
(\dot{Z}_y C_y)^*
\end{bmatrix}$, $\Omega_p := \begin{bmatrix}
(E + B_H F)^* & 0 \\
0 & I
\end{bmatrix}$, and $C_p := -(B_H Z)^* 0$. Now

$$
\dot{H}_{\infty}^\sim(z) \dot{N}_v(z) \dot{N}_v^\sim(z) - \dot{V}(z) = J_0^+ \begin{bmatrix}
A_p \\
C_p
\end{bmatrix} \begin{bmatrix}
B_p \\
0
\end{bmatrix} \begin{bmatrix}
[z \Omega_p - I]
\end{bmatrix}
$$

Now, we have all component to write the optimal relaxed causal sampler described in Proposition 3.
Theorem 2 Let system $\hat{G}$ and $\hat{H}$ are given by STPBC (33) and (34) respectively. Let assumptions $A_3 - A_8$ be satisfied. If the conditions given in Theorem 1 are satisfied then

$$\dot{S}_{\text{opt}} := \arg\inf_{S \in \mathbb{Z}^H} \| \dot{G}_v - \hat{H} \dot{S}_{\text{opt}} \hat{G}_y \|_{L^2} = \hat{H}_o^{-1}(\dot{S}_{\text{opt}} \hat{M}_y - \hat{M}_h)$$

where $\dot{S}_{\text{opt}} := \text{proj}_{L^2}(\dot{H}_o^{-1} \dot{N}_v \dot{N}_v^\top - \dot{V})$ has STPBC

$$\dot{S}_{\text{opt}}(z) = \mathcal{J}_0^+ \left[ \frac{A_p}{C_p \left( I - \left( z e^{-A_p \Omega_p} \right)^{l+1} \right) B_p} \right] [z \Omega_p - I]$$

where $A_p$, $B_p$, $\Omega_p$, and $C_p$ are defined in Corollary 2.

8.8 Optimal $H^2$ norm

In this section we provide a simple algebraic expression to calculate the optimal error norm $\| \dot{G}_{e, \text{opt}} \|_{L^2}$ for signal generator $\dot{G}$ given in (32) and hold $\dot{H}$ given in (34). It is given in (14) that

$$\| \dot{G}_{e, \text{opt}} \|_{L^2} := \| \dot{G}_v - \hat{H} \dot{S}_{\text{opt}} \dot{G}_y \|_{L^2}^2 = \| \dot{G}_v + \hat{H}_o \hat{M}_h \dot{G}_y \|_{L^2}^2 - \| \dot{S}_{\text{opt}} \|_{L^2}^2$$

where $\dot{S}_{\text{opt}}$ is given in Theorem 2 and

$$\| \dot{S}_{\text{opt}} \|_{L^2}^2 = \| \text{proj}_{L^2}(\dot{H}_o^{-1} \dot{N}_v \dot{N}_v^\top - \dot{V}) \|_{L^2}^2$$

The squared $L^2$ norm of $\dot{S}_{\text{opt}}$ can also be written as

$$\| \dot{S}_{\text{opt}} \|_{L^2}^2 = \| \dot{H}_o^{-1} \dot{N}_v \dot{N}_v^\top - \dot{V} \|_{L^2}^2 - \| \text{proj}_{L^2 \setminus L^2}(\dot{H}_o^{-1} \dot{N}_v \dot{N}_v^\top - \dot{V}) \|_{L^2}^2$$

It is shown later that it is easier to calculate $\| \dot{S}_{\text{opt}} \|_{L^2}$ using (50) than (49).

Now, we obtain STPBC of all systems required to calculate $\| \dot{G}_{e, \text{opt}} \|_{L^2}$. Using Corollary 2, the STPBC of sampler $\dot{H}_o^{-1} \dot{N}_v \dot{N}_v^\top - \dot{V}$ is given by

$$\dot{H}_o(z) \dot{N}_v(z) \dot{N}_v^\top(z) - \dot{V}(z) = \mathcal{J}_0^+ \left[ \frac{A_p}{C_p \left( z e^{-A_p \Omega_p} \right)^{l+1} B_p} \right] [z \Omega_p - I]$$

where $A_p$, $B_p$, $\Omega_p$, and $C_p$ are defined in Corollary 2. Proceeding as in the proof Theorem 2, we have

$$\text{proj}_{L^2 \setminus L^2}(\dot{H}_o^{-1} \dot{N}_v \dot{N}_v^\top - \dot{V}) := \mathcal{J}_0^+ \left[ \frac{A_p}{C_p \left( z e^{-A_p \Omega_p} \right)^{l+1} B_p} \right] [z \Omega_p - I]$$

Note that $\dot{H}_1$ is a sampler and $\dot{M}_h$ is a sampler, therefore to obtain the STPBC of $\dot{G}_v + \dot{H}_1 \dot{M}_h \dot{G}_y$, we need the STPBC of $\dot{H}_1 \dot{M}_h$. Using [17, lemma 3], (41) and (48), the STPBC of $\dot{H}_1 \dot{M}_h$ is given by

$$\dot{H}_1(z) \dot{M}_h(z) = \frac{A_H}{C_H} \begin{bmatrix} 0 & 0 & A + LC_y & L \\ 0 & 0 & 0 & 0 \\ z [I \ M_1] & [(E + B_H F) \ 0] \\ 0 & 0 & 0 & I \end{bmatrix}$$

$$= \frac{A_H}{C_H} \begin{bmatrix} 0 & 0 & A + LC_y & L \\ 0 & 0 & 0 & 0 \\ z [I \ 0] & [(E + B_H F) \ -M_1] \end{bmatrix}$$

where

$$M_1 := -B_H Z (B_H Z)^* P_0.$$
Lemma 19 Let the STPBC of $\hat{G} = \begin{bmatrix} G_v \\ G_y \end{bmatrix}$, $\hat{M}_h$ and $\hat{H}_1$ are given by (32), (48) and (41) respectively. Then, the STPBC of the system $\hat{G}_v + \hat{H}_1 \hat{M}_h \hat{G}_y$ is given by

$$\hat{G}_v(z) + \hat{H}_1(z)\hat{M}_h(z)\hat{G}_y(z) = \begin{bmatrix} A_e & B_e \\ C_e & 0 \end{bmatrix} \begin{bmatrix} zI - Y_c \end{bmatrix}$$

(55)

where

$$A_e := \begin{bmatrix} A_H & 0 \\ 0 & A + L C_y \end{bmatrix}, \quad B_e := \begin{bmatrix} 0 \\ B + L D_y \end{bmatrix}$$

$$C_e := [C_H \ 0 \ C_v], \quad Y_c := \begin{bmatrix} E + B_H F - M_1 & M_1 \\ 0 & I \end{bmatrix}.$$  

Since $\hat{G}_v + \hat{H}_1 \hat{M}_h \hat{G}_y$ is causal and in $L^\infty$, it is in $H^\infty$. By Lemma 5, this further implies $\hat{G}_v + \hat{H}_1 \hat{M}_h \hat{G}_y \in H^2$. However, in the following theorem we show that the STPBC of the system $\hat{G}_v + \hat{H}_1 \hat{M}_h \hat{G}_y$ given in (55) is not minimal. It contains unobservable or uncontrollable poles that lie outside (open) unit disc of the complex plane. Since $\hat{G}_v + \hat{H}_1 \hat{M}_h \hat{G}_y \in H^\infty$, this implies these poles must be canceled somehow. This complicates the calculation of $H^2$ norm of $\hat{G}_v + \hat{H}_1 \hat{M}_h \hat{G}_y$ as shown in the following lemma.

Lemma 20 Let $\hat{G}_v$ be given by STPBC (33). Let the matrix functions $\Lambda$ and $\Gamma$ be as in Lemma 8. Also let $A_e$, $B_e$, $\Upsilon_c$ and $C_e$ be as in (55). Let $\hat{C}_e$ and $\hat{B}_e$ be any matrices which satisfy

$$\hat{C}_e \hat{C}_e = \int_0^h e^{A_s} \hat{C}_e \hat{C}_e e^{A_s} ds = \Lambda_{22}(A_e, C_e) A_{12}(A_e, C_e)$$

(56a)

$$\hat{B}_e \hat{B}_e = \int_0^h e^{A_s} \hat{B}_e \hat{B}_e e^{A_s} ds = \Gamma_{33}(A_e, B_e) \Gamma_{13}(A_e, B_e)$$

(56b)

respectively. Define for $\tau \in [0, h)$

$$\hat{D}_e \hat{u} := \int_0^h C_e e^{A_e(\tau - \sigma)} B_e \Lambda(\tau - \sigma) \hat{u}(\sigma) d\sigma$$

(57)

then

$$||\hat{D}_e||^2_{HS} = \text{tr} C_e \int_0^h \int_0^h e^{A_s} \hat{B}_e \hat{B}_e e^{A_s} ds dt \ C_e^*$$

$$= \text{tr} C_e \Gamma_{13}(A_e, B_e) \Gamma_{33}(A_e, B_e) C_e^*.$$  

(58)

Let us partition $M_1$ defined in (54) as $[M_{1s} \ M_{1u}]$ according to $A_s$ and $A_u$, and define

$$A_{ms} := \begin{bmatrix} (E + B_H F) e^{A_A h} - M_1 e^{(A + L C_y) h} & M_1 e^{A_A h} \\ 0 & e^{(A + L C_y) h} \end{bmatrix}.$$  

$$A_{ms} := \begin{bmatrix} (E + B_H F) e^{A_A h} - M_1 e^{(A + L C_y) h} & M_1 e^{A_A h} \\ 0 & e^{(A + L C_y) h} \end{bmatrix}.$$  

Also let us partition

$$\Upsilon_e \hat{B}_e =: \begin{bmatrix} \hat{B}_{ms} \\ \hat{B}_{mu} \end{bmatrix}, \quad \hat{C}_e =: \begin{bmatrix} \hat{C}_{ms} & \hat{C}_{mu} \end{bmatrix}$$

according to $A_{ms}$ and $e^{A_A h}$. Then

$$||\hat{G}_v + \hat{H}_1 \hat{M}_h \hat{G}_y||^2_{HS} = \frac{1}{h} ||\hat{D}_e||^2_{HS} + \frac{1}{h} \text{tr}(C_{ms} C_{ms} W_{ec})$$

(59)

$$= \frac{1}{h} ||\hat{D}_e||^2_{HS} + \frac{1}{h} \text{tr}(W_{eo}(\hat{B}_{ms} - X_m \hat{B}_{mu})(\hat{B}_{ms} - X_m \hat{B}_{mu})^*).$$

(60)

where $W_{ec}$ and $W_{eo}$ are matrices satisfying the Lyapunov equations

$$W_{ec} = A_{ms} W_{ec} A_{ms} + (\hat{B}_{ms} - X_m \hat{B}_{mu})(\hat{B}_{ms} - X_m \hat{B}_{mu})^*,$$
The proof of the above theorem follows from lemmas Optimal relaxed causal sampling from system theoretic viewpoint 19 and X in H are obtained in Lemma 3. Hence, their H^2 norm can be obtained by Lemma 7. Since \|z_i \bar{G}\|^2 = \|\bar{G}\|^2 for a system \bar{G}, L^2 norm of the system proj_{L^2}(H_i^\sim \hat{N}_v \hat{N}_y^\sim - \hat{V}) can be calculated easily.

Lemma 21 Let A_p, B_p, \rho_p, and C_p be as in Corollary 2. Let the STPBCs of the systems \hat{H}_i^\sim \hat{N}_v \hat{N}_y^\sim - \hat{V} and proj_{L^2}(H_i^\sim \hat{N}_v \hat{N}_y^\sim - \hat{V}) be given by (51) and (52) respectively. Then

1. the L^2 norm of (anti-causal) sampler \hat{H}_i^\sim \hat{N}_v \hat{N}_y^\sim - \hat{V} is given by

$$\|\hat{H}_i^\sim \hat{N}_v \hat{N}_y^\sim - \hat{V}\|^2_{L^2} = \frac{1}{h} \text{tr}(B_p \hat{B}_p W_{sc}) = \frac{1}{h} \text{tr}(W_{sc} \bar{C}_p^2)$$

where W_{sc} and W_{so} are matrices satisfying the Lyapunov equations

$$W_{sc} = \Omega_p e^{-A_p h} W_{sc} e^{-A_p h} \Omega_p + \bar{C}_p \bar{C}_p,$$

$$W_{so} = e^{-A_p h} \Omega_p W_{so} \Omega_p^* e^{-A_p h} + \hat{B}_p \hat{B}_p^*$$

and \hat{B}_p is any matrix satisfying

$$\hat{B}_p \hat{B}_p^* = \int_0^h e^{-A_p \tau} B_p(\tau) B_p(\tau)^* e^{-A_p \tau} d\tau = P_2 A_z^2(A_z^* B_z^*) A_{12}(A_z^* B_z^*) P_2^*$$

in which

$$A_z := \begin{bmatrix} A_H C_H C_v X C_H^* C_v \end{bmatrix}, B_z := \begin{bmatrix} 0 \end{bmatrix}, P_2 := \begin{bmatrix} I \text{ } \bar{C}_p \end{bmatrix}$$

2. the L^2 norm of (anti-causal) sampler proj_{L^2}(H_i^\sim \hat{N}_v \hat{N}_y^\sim - \hat{V}) is given by

$$\|\text{proj}_{L^2}(H_i^\sim \hat{N}_v \hat{N}_y^\sim - \hat{V})\|^2_{L^2} = \frac{1}{h} \text{tr}(\hat{B}_p \hat{B}_p^* W_{pc}) = \frac{1}{h} \text{tr}(W_{pc} C_{pm} C_{pm})$$

where C_{pm} := C_p(e^{-A_p h} \Omega_p)^{t+1} and W_{pc} and W_{po} are matrices satisfying the Lyapunov equations

$$W_{pc} = \Omega_p e^{-A_p h} W_{pc} e^{-A_p h} \Omega_p + \bar{C}_{pm} \bar{C}_{pm},$$

$$W_{po} = e^{-A_p h} \Omega_p W_{po} \Omega_p^* e^{-A_p h} + \hat{B}_p \hat{B}_p^*$$

Now we summarize all of the results in this section to obtain the optimal error norm ||\bar{G}_{e, opt}||_{L^2}.

Theorem 3 Let system \bar{G} and \hat{H} are given by STPBC (33) and (34) respectively. Let assumptions A_9 - A_8 be satisfied. If the conditions given in Theorem 1 are satisfied then the optimal error norm ||\bar{G}_{e, opt}||_{L^2} is given by

$$||\bar{G}_{e, opt}||_{L^2} = \bar{G}_v + \hat{H}_i \bar{M}_h \bar{G}_y ||_{L^2} - ||\hat{H}_i^\sim \hat{N}_v \hat{N}_y^\sim - \hat{V}||_{L^2}$$

where ||\bar{G}_v + \hat{H}_i \bar{M}_h \bar{G}_y||_{L^2} is obtained in Lemma 20, and ||\hat{H}_i^\sim \hat{N}_v \hat{N}_y^\sim - \hat{V}||_{L^2} and ||\text{proj}_{L^2}(H_i^\sim \hat{N}_v \hat{N}_y^\sim - \hat{V})||_{L^2} are obtained in Lemma 21.

The proof of the above theorem follows from lemmas 11, 15, 20 and 21.
8.9 Example

In this section we consider an example to explain the theory discussed till now.

Example 1 Consider LCTI systems $G_v$ and $G_y$ given in the Laplace domain by $G_v(s) = \left[ \frac{1}{\tau}, 0 \right]$ and $G_y(s) = \left[ \frac{1}{\tau}, \epsilon \right]$ where $\epsilon > 0$.

The STPBC of $G_v$ and $G_y$ is given by

\[
\begin{bmatrix}
G_v(z) \\
\dot{G}_y(z)
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
0 & \epsilon
\end{bmatrix}
\begin{bmatrix}
z & -1
\end{bmatrix}.
\]

We also assume that the hold is the ideal zero order hold given by

\[
\dot{H}_{iz}(z) = \begin{bmatrix}
0 \\
1
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
[1] & 0
\end{bmatrix}J_{0}^{+}.
\]

In this example we obtain strictly causal, causal, 1-causal and non-causal optimal samplers given the systems $\dot{G}_v$, $\dot{G}_y$ and $\dot{H}_{iz}$.

The first thing we need to check is the condition $(I - \dot{H}_i \dot{H}_i^\sim) \dot{G}_v \in L^\infty$ (see Proposition 1) for the existence of a solution of Problem 2 where $\dot{H}_i$ is the inner factor of hold $\dot{H}_{iz}$. In Theorem 1, we saw that this condition can be checked without constructing the inner factor $\dot{H}_i$. In this example, $C_{vu} = C_H = \sqrt{\eta}$, $P_a = h$, $E = 0$, $D^+ = \frac{1}{\sqrt{\eta}}$ and $D^- = 0$. As required in Theorem 1,

\[
\dot{C}_{vu}^* C_{vu} - P_v (\dot{C}_H^* \dot{C}_H)^+ P_a = h - h = 0
\]

and $X_1 = 0$ satisfies (44).

Using Lemma 15, we have $F = 0$, $Z = \frac{1}{\sqrt{\eta}}$ and an inner-outer factorization of $\dot{H}_{ix} := \dot{H}_i \dot{H}_o$ is given by

\[
\dot{H}_{ix}(z) = \frac{1}{\sqrt{\eta}} \begin{bmatrix}
0 & 1 \\
[1] & 0
\end{bmatrix}J_{0}^{+}, \quad \dot{H}_o(z) = \sqrt{\eta}
\]

And using Lemma 11, we have $Z_y = \frac{1}{\epsilon}$, $X = \epsilon$, and $L = -\frac{1}{\epsilon}$ and

\[
\begin{bmatrix}
\dot{M}_v(z) \\
\dot{M}_y(z)
\end{bmatrix} = \begin{bmatrix}
-\frac{2}{\epsilon} & 1 \\
1 & \epsilon
\end{bmatrix}
\begin{bmatrix}
z & -1
\end{bmatrix}.
\]

Using Lemma 17, $P(t) = h - t$. From (45) and (48), we have

\[
\dot{V}(z) = J_{0}^{+} \begin{bmatrix}
0 & -P(\tau) \\
-\frac{1}{\sqrt{\eta}} & 0
\end{bmatrix}
\begin{bmatrix}
0 & -I
\end{bmatrix}
\]

\[
\dot{M}_{h}(z) = J_{0}^{+} \begin{bmatrix}
-\frac{2}{\epsilon} & 1 \\
-\frac{1}{\sqrt{\eta}} & 0
\end{bmatrix}
\begin{bmatrix}
z & -1
\end{bmatrix}.
\]

Using Corollary 2, we have

\[
\dot{H}_i^\sim(z) \dot{N}_v(z) \dot{N}_v^\sim(z) - \dot{V}(z) = J_{0}^{+} \begin{bmatrix}
A_p \\
C_p
\end{bmatrix}
\begin{bmatrix}
B_p \\
0
\end{bmatrix}
\begin{bmatrix}
[z \Omega_p - I]
\end{bmatrix}
\]

where $A_p := \begin{bmatrix}
0 & -\epsilon \\
\frac{1}{\tau} & 0
\end{bmatrix}$, $B_p := \left[ P(\tau) \right]$, $\Omega_p := \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}$, and $C_p := \left[ \frac{1}{\sqrt{\eta}} \right]$.

Using Theorem 2, $\dot{S}_{a,\text{opt}} := \text{proj}_{HZ}(\dot{H}_i^\sim \dot{N}_v \dot{N}_v^\sim - \dot{V})$ is given by STPBC

\[
\dot{S}_{a,\text{opt}}(z) = J_{0}^{+} \begin{bmatrix}
A_p \\
C_p
\end{bmatrix}
\begin{bmatrix}
I - (ze^{-A_p \kappa} \Omega_p)^+ \\
0
\end{bmatrix}
\begin{bmatrix}
B_p \\
0
\end{bmatrix}
\begin{bmatrix}
[z \Omega_p - I]
\end{bmatrix}
\]

and the optimal relaxed causal sampler $\hat{S}_{a,\text{opt}}$ is given by (12).

Now, we write strictly causal, causal, 1-causal and non-causal optimal samplers in a more tangible form.
1. Strictly causal optimization (i.e. \( l = -1 \))

Since \( \text{proj}_{\mathbf{H}^2} \{ \hat{H}_r^- \hat{N}_v \hat{N}_y^- - \hat{V} \} = 0 \), we have strictly causal sampler \( -\hat{H}_o^{-1} \hat{M}_h \) which is a cascade of the LCTI causal system \( 1/(\epsilon s + 1) \) and the ideal sampler. Note that the strictly causal sampler does not depend upon \( h \) and tend to a causal impulse when \( \epsilon \to 0 \). Therefore, strictly causal \( \hat{S}_{\text{opt}} \) in time domain is a sampler with sampling function \( \psi_{-1}(t) := \frac{1}{h} e^{-\frac{t}{h}} [0, \infty) (t) \). See Figure 3.

2. Causal optimization (i.e. \( l = 0 \))

Since \( \hat{H}_r^- \in \mathbf{H}^\infty \), therefore \( \hat{V} = 0 \) satisfies \( \hat{H}_r^- \hat{M}_v - \hat{V} \hat{M}_y \in \mathbf{H}^\infty \). Therefore,

\[
\hat{S}_{\text{opt}} = \text{proj}_{\mathbf{H}^2}(\hat{H}_r^- \hat{N}_v \hat{N}_y^-).
\]

We find that causal \( \hat{S}_{\text{opt}} \) in time domain is a sampler with sampling function

\[
\psi_0(t) := \begin{cases} 
0 & t \leq -h \\
-\frac{1}{h} e^{-\frac{t}{h}} - \frac{1}{2} e^{-\frac{h}{2}} & -h < t \leq 0 \\
\frac{1}{2} e^{-\frac{t}{h}} - \frac{1}{2} e^{-\frac{h}{2}} & t > 0
\end{cases}
\]

See Figure 3.

3. 1-causal optimization

Similar to causal optimization, we take \( \hat{V} = 0 \). Therefore,

\[
\hat{S}_{\text{opt}} = \text{proj}_{\mathbf{H}^2}(\hat{H}_r^- \hat{N}_v \hat{N}_y^-).
\]

We find that 1-causal \( \hat{S}_{\text{opt}} \) in time domain is a sampler with sampling function

\[
\psi_1(t) := \begin{cases} 
0 & t \leq -2h \\
\frac{1}{2}(-e^{-\frac{t}{h}} + e^{-\frac{h}{2}} - e^{-\frac{2h}{3}} + e^{-\frac{5h}{3}}) & -2h < t \leq -h \\
\frac{1}{2}(-e^{-\frac{t}{h}} - e^{-\frac{h}{2}} - e^{-\frac{2h}{3}} + e^{-\frac{5h}{3}}) + 1 & -h < t \leq 0 \\
\frac{1}{2}(e^{-\frac{t}{h}} - e^{-\frac{h}{2}} - e^{-\frac{2h}{3}} + e^{-\frac{5h}{3}}) & t > 0
\end{cases}
\]

See Figure 3.

4. Non-causal optimization (i.e. \( l = \infty \))

In this case,

\[
\hat{S}_{\text{opt}} = \hat{H}_r^- \hat{N}_v \hat{N}_y^-.
\]

We find that non-causal \( \hat{S}_{\text{opt}} \) in time domain is a sampler with sampling function

\[
\psi_\infty(t) := \begin{cases} 
-\frac{1}{2} e^{-\frac{t}{h}} + \frac{1}{2} e^{-\frac{h}{2}} & t \leq -h \\
-\frac{1}{2} e^{-\frac{t}{h}} - \frac{1}{2} e^{-\frac{h}{2}} + 1 & -h < t \leq 0 \\
\frac{1}{2} e^{-\frac{t}{h}} - \frac{1}{2} e^{-\frac{h}{2}} & t > 0
\end{cases}
\]

See Figure 3.

If \( \epsilon \to 0 \) in this example, then \( \psi_0(t), \psi_1(t), \psi_\infty(t) \) converge to the following

\[
\psi_{\infty}(t) = \begin{cases} 
1/h & -h < t \leq 0 \\
0 & \text{elsewhere}
\end{cases}
\]

see Figure 4.

Figure 5 shows the optimal error \( \| \mathcal{G}_{\epsilon, \text{opt}} \|_{L^2} := \| \mathcal{G}_v - \mathcal{H} \hat{S}_{\text{opt}} \mathcal{G}_y \|_{L^2} \) for different optimal relaxed causal samplers. As expected the optimal error decreases with increasing non-causality.
Fig. 4 Sampling function of optimal $l$-causal (where $l \geq 0$) sampler in Example 1 when $\epsilon \to 0$.

Fig. 5 Optimal error $\|G_{e, \text{opt}}\|_{L^2} := \|G_{v} - H_{\text{opt}}G_{y}\|_{L^2}$ for different $l$ and $\epsilon = 0.5$.

9 Conclusions

In this paper we obtained a stable and optimal $l$-causal sampler given hold and a LCTI model $G$. The presence of hold $H$ complicates the question of existence of such a sampler when $G$ is unstable. We also provided the conditions of existence of optimal $l$-causal samplers, in (lifted) frequency domain as well as in state space with two point boundary condition (STPBC). We also gave the optimal $l$-causal sampler in STPBC. Use of STPBC allows an easy and clear framework to solve our problem.

10 Acknowledgment

We are thankful to Prof. Leonid Mirkin (Technion, Israel) for many help discussions and suggestion (esp. on Remark 4).

11 Appendices

11.1 Proof of the results in Section 6

Proof (Proposition 1): We first prove that these three criteria are necessary. Let us assume that there exists an $\tilde{S} \in z^l H^\infty$ such that $\tilde{G}_e := \tilde{G}_v - H \tilde{S} \tilde{G}_y \in L^\infty$.

Condition 1: Since $R := \begin{bmatrix} \bar{H}^- \\ I - \bar{H}^-H_{o}^- \end{bmatrix}$ satisfies $R^-R = I$, we have

$$\|\tilde{G}_e\|_{L^\infty} = \|\tilde{G}_v - \bar{H}^- \tilde{S} \tilde{G}_y\|_{L^\infty} = \|\bar{R} \tilde{G}_v - R \bar{H} H_{o} \bar{S} \tilde{G}_y\|_{L^\infty}$$

$$= \left\| \begin{bmatrix} \bar{H}^- \tilde{G}_v - \bar{H}_{o} \bar{S} \tilde{G}_y \\ \bar{H}_{o} \bar{G}_v \end{bmatrix} \right\|_{L^\infty}$$

By assumption $\tilde{G}_e \in L^\infty$, so $\|\bar{H}_{o} \bar{G}_v\|_{L^\infty}$ is finite.

Condition 2: Since the hold $H$ is stable and casual, we have that $\tilde{K} := \bar{H} \bar{S} \in z^l H^\infty$ and $\tilde{G}_v - \tilde{K} \tilde{G}_y \in L^\infty$. According to [21, proposition 2.1], the existence of a $\tilde{K} \in z^l H^\infty$ that renders $\tilde{G}_v - \tilde{K} \tilde{G}_y \in L^\infty$ is equivalent
to existence of a factorization over $H^\infty$ of $\tilde{G}$ of the form (9) with $\tilde{M}_y, \tilde{N}_y$ left coprime. Note that the coprime factorization is over $H^\infty$ not $z'H^\infty$ as we expect. This is related to the causality of $\tilde{G}$ (see [12, remark 20.4] for detail).

**Condition 3:** Let $\hat{V} := (\hat{H}_\hat{\tilde{M}}v + \hat{H}_o\hat{S})\hat{M}_y^{-1}$. Then, $\hat{H}_\hat{\tilde{M}}v - \hat{V}\hat{M}_y = -\hat{H}_o\hat{S} \in z'H^\infty$ as required. Hence,

$$\hat{V}\hat{M}_y = \hat{H}_\hat{\tilde{M}}v + \hat{H}_o\hat{S} \in L^\infty.$$ 

Also,

$$\hat{V}\hat{N}_y = (\hat{H}_\hat{\tilde{M}}v + \hat{H}_o\hat{S})\hat{G}_y \in L^\infty$$

because it follows from (62) that

$$\hat{H}_\hat{\tilde{M}}v - \hat{H}_o\hat{S}\hat{G}_y \in L^\infty$$

$$\Rightarrow \hat{H}_\hat{\tilde{M}}v - \hat{H}_\hat{\tilde{M}}v\hat{G}_y - \hat{H}_o\hat{S}\hat{G}_y \in L^\infty$$

$$\Rightarrow \hat{H}_\hat{\tilde{M}}v\hat{G}_y + \hat{H}_o\hat{S}\hat{G}_y \in L^\infty,$$ 

as $\hat{H}_\hat{\tilde{M}}v \hat{G}_y \in L^\infty$.

Here, we used $\hat{G}_y = \hat{N}_y - \hat{M}_y\hat{G}_y$ and $\hat{G}_y = \hat{M}_y^{-1}\hat{N}_y$ which follows from (9). Therefore, we have

$$\hat{V} [\hat{N}_y \hat{M}_y] \in L^\infty$$

This further implies that $\hat{V} \in L^\infty$ as $[\hat{N}_y \hat{M}_y]$ is right invertible in $H^\infty$.

Now assume that Conditions 1-3 are satisfied. We show that there exists a sampler $\hat{S} \in z'H^\infty$ such that $\tilde{G}_o \in L^\infty$. From (62), a sampler $\hat{S} \in z'H^\infty$ achieves $\tilde{G}_o \in L^\infty$ iff $\|H\hat{G}_v\|_{L^\infty} < \infty$ and $\|H\hat{G}_v - H_o\hat{S}\hat{G}_y\|_{L^\infty} < \infty$. From Condition 2 and 3, it is clear that we have a $\hat{V} \in L^\infty$ such that $M_h := \hat{H}_\hat{\tilde{M}}v - \hat{V}\hat{M}_y \in z'H^\infty$ and $\hat{G}_v = \hat{N}_v - \hat{M}_v\hat{G}_y$ and $\hat{G}_y = \hat{M}_y^{-1}\hat{N}_y$. Now, $\hat{S} := -\hat{H}_\hat{\tilde{M}}v\hat{M}_h$ does the job because $\hat{S} \in z'H^\infty$ and

$$\hat{H}_\hat{\tilde{M}}v - \hat{H}_o\hat{S}\hat{G}_y = \hat{H}_\hat{\tilde{M}}v\hat{G}_y + (-\hat{H}_\hat{\tilde{M}}v + \hat{M}_h)\hat{G}_y$$

$$= \hat{H}_\hat{\tilde{M}}v\hat{G}_y - \hat{V}\hat{M}_y\hat{G}_y = \hat{H}_\hat{\tilde{M}}v\hat{N}_v - \hat{V}\hat{N}_y = \hat{V}\hat{N}_y \in L^\infty.$$

$$\square$$

**Proof (Lemma 1):** We have

$$[\hat{G}_y \hat{I}] = \hat{M}_y^{-1}[\hat{N}_y \hat{M}_y]$$

The right invertibility of $[\hat{N}_y \hat{M}_y]$ implies that $\hat{M}_y^{-1}$ is causal if $\hat{G}_v$ is causal.

Now, it is given that $\hat{H}_\hat{\tilde{M}}v\hat{M}_y - \hat{V}_1\hat{M}_y \in z'H^\infty$ and $\hat{H}_\hat{\tilde{M}}v\hat{M}_y - \hat{V}_2\hat{M}_y \in z'H^\infty$. This implies that $(\hat{V}_2 - \hat{V}_1)\hat{M}_y \in z'H^\infty$ and so that $\hat{V}_2 - \hat{V}_1$ is $l$-causal as $\hat{M}_y^{-1}$ is causal.

Since $\hat{V}_1$ and $\hat{V}_2$ are in $L^\infty$ and are samplers, this implies $\hat{V}_2 - \hat{V}_1 \in L^2$ [13]. So, $\hat{V}_2 - \hat{V}_1 \in L^2$ and $l$-causal. It means $\hat{V}_2 - \hat{V}_1 \in z'H^2$. Hence, $\proj_{z'H^2}(\hat{V}_1 - \hat{V}_2) = 0$.

**Proof (Proposition 2):** Given condition 1 of Proposition 1, it follows from Equation (62) that $\|\hat{G}_o\|_{L^\infty}$ is finite iff $\hat{G}_o := \hat{H}_\hat{\tilde{M}}v - H_o\hat{S}\hat{G}_y$ is in $L^\infty$.

Given condition 1-3 of Proposition 1, $\hat{N}_o := \hat{H}_\hat{\tilde{M}}v\hat{N}_v - \hat{V}\hat{N}_y \in L^\infty$. We now show that every solution $\hat{S} \in z'H^\infty$ has the form (10). Let $\hat{S} \in z'H^\infty$ be such that $\hat{G}_o \in L^\infty$. This means $\hat{G}_o \in z'H^\infty$ as $\hat{G}_v, \hat{G}_y$ are causal and $\hat{S}$ is $l$-causal. Let $\hat{G}_{oh} := \hat{H}_\hat{\tilde{M}}v - H_o\hat{S}\hat{G}_y$. Clearly $\hat{G}_{oh} \in L^\infty$ and $\hat{H}_\hat{\tilde{M}}v\hat{G}_v = \hat{H}_\hat{\tilde{M}}v\hat{N}_v - \hat{H}_\hat{\tilde{M}}v\hat{M}_y\hat{G}_y = \hat{N}_o - \hat{M}_h\hat{G}_y$ by Proposition 1. Therefore,

$$\hat{G}_{oh} = \hat{H}_\hat{\tilde{M}}v\hat{G}_v - H_o\hat{S}\hat{G}_y$$

$$\hat{G}_{oh} - \hat{N}_o = -\hat{M}_h\hat{G}_y - H_o\hat{S}\hat{G}_y$$

Since $-\hat{M}_h\hat{G}_y - H_o\hat{S}\hat{G}_y$ is $l$-causal as $(\hat{M}_h + H_o\hat{S} \in z'H^\infty)$ and $\hat{G}_{oh} - \hat{N}_o \in L^\infty$, we have $\hat{G}_{oh} - \hat{N}_o \in z'H^\infty$.

Hence,

$$\hat{M}_h\hat{G}_y + H_o\hat{S}\hat{G}_y \in z'H^\infty$$
Using $\hat{G}_y = \hat{M}_y^{-1} \hat{N}_y$ and $\hat{M}_h + \hat{H}_o \hat{S} \in z^1 H^\infty$, the above implies

$$(\hat{M}_h + \hat{H}_o \hat{S}) G_y I = (\hat{M}_h + \hat{H}_o \hat{S}) \hat{M}_y^{-1} \begin{bmatrix} \hat{N}_y & \hat{M}_y \end{bmatrix} \in z^1 H^\infty$$

This implies $(\hat{M}_h + \hat{H}_o \hat{S}) \hat{M}_y^{-1} \in z^1 H^\infty$ as $\begin{bmatrix} \hat{N}_y & \hat{M}_y \end{bmatrix}$ is right invertible in $H^\infty$. Hence, we have that the $\hat{S}_\alpha$ defined as

$$\hat{S}_\alpha := (\hat{M}_h + \hat{H}_o \hat{S}) \hat{M}_y^{-1}$$

is in $z^1 H^\infty$. From this $\hat{S}_\alpha$, the sampler $\hat{S}$ follows as

$$\hat{S} = \hat{H}_o^{-1}(-\hat{M}_h + \hat{S}_\alpha \hat{M}_y)$$

On the other hand, if $\hat{S} := \hat{H}_o^{-1}(\hat{S}_\alpha \hat{M}_y - \hat{M}_h)$ where $\hat{S}_\alpha \in z^1 H^\infty$ then clearly $\hat{S} \in z^1 H^\infty$, $\hat{G}_e := \hat{H}_o^{-1} \hat{G}_o - \hat{H}_o \hat{S} \hat{G}_y = \hat{N}_h - \hat{S}_\alpha \hat{N}_y \in L^\infty$ and $\hat{G}_e \in L^\infty$ (given all conditions of Proposition 1 are satisfied). In the end using $\hat{S} := \hat{H}_o^{-1}(\hat{S}_\alpha \hat{M}_y - \hat{M}_h)$, we have

$$\hat{G}_e = \hat{G}_v - \hat{H} \hat{S} \hat{G}_y = \hat{G}_v - \hat{H}(\hat{S}_\alpha \hat{M}_y - \hat{M}_h)\hat{G}_y$$

$$= \hat{G}_v + \hat{H}_o \hat{M}_y \hat{G}_y - \hat{H} \hat{S}_\alpha \hat{N}_y$$

Note that $\hat{M}_h := \hat{H}_o^{-1} \hat{M}_o - \hat{V} \hat{M}_y$ depends upon $\hat{V}$ which is not fixed. Therefore, $\hat{S}$ and $\hat{G}_e$ depend upon $\hat{V}$ and $\hat{S}_\alpha$. This may (wrongly) suggests that the parameterization of $\hat{S}$ and $\hat{G}_e$ given in (10) and (11) is in two parameters. However, we will show now that the parameterization of $\hat{S}$ and $\hat{G}_e$ is in single parameter. Any $\hat{V} \in L^\infty$ (also implies $\hat{V} \in L^2$ [13]) satisfying $\hat{M}_h := \hat{H}_o^{-1} \hat{M}_o - \hat{V} \hat{M}_y \in z^1 H^\infty$ can be represented as $\hat{V} = \text{proj}_{L^2 \setminus z^1 H^2} \hat{V} + \text{proj}_{z^1 H^2} \hat{V}$. Therefore,

$$\hat{S} := \hat{H}_o^{-1}(\hat{S}_\alpha \hat{M}_y - \hat{M}_h)$$

$$= \hat{H}_o^{-1} \left( \hat{S}_\alpha \hat{M}_y - \hat{H}_o^{-1} \hat{M}_o + \hat{V} \hat{M}_y \right)$$

$$= \hat{H}_o^{-1} \left( \hat{S}_\alpha \hat{M}_y - \hat{H}_o^{-1} \hat{M}_o + (\text{proj}_{L^2 \setminus z^1 H^2} \hat{V} + \text{proj}_{z^1 H^2} \hat{V}) \hat{M}_y \right)$$

$$= \hat{H}_o^{-1} \left( (\hat{S}_\alpha + \text{proj}_{L^2 \setminus z^1 H^2} \hat{V}) \hat{M}_y - \hat{H}_o^{-1} \hat{M}_o + \text{proj}_{L^2 \setminus z^1 H^2} \hat{V} \hat{M}_y \right)$$

Since $\text{proj}_{L^2 \setminus z^1 H^2} \hat{V}$ is unique by Lemma 1, $\hat{S}$ is still parameterized by a new single parameter $\hat{S}_\alpha + \text{proj}_{z^1 H^2} \hat{V} \in z^1 H^\infty$.

Proof (Proposition 3) : $\hat{N}_y \hat{N}_y^\sim$ is stable (by construction) and rational (Assumption $A_1$). Also $\hat{N}_y \hat{N}_y^\sim$ has no unit circle zeros (Assumption $A_3$). Therefore, $\hat{N}_y \hat{N}_y^\sim$ has a spectral co-factorization $\hat{W} \hat{W}^\sim$ where $\hat{W}$ is bistable and bicausal in $H^\infty$ (see e.g. [29]). This means $\hat{W}^{-1} \hat{N}_y$ is co-inner. As $\hat{G}_y = \hat{M}_y^{-1} \hat{N}_y = (\hat{W}^{-1} \hat{M}_y)^{-1}(\hat{W}^{-1} \hat{N}_y)$, we have a coprime factorization of $\hat{G}_y$ with coprime factors $\hat{W}^{-1} \hat{M}_y$ and co-inner $\hat{W}^{-1} \hat{N}_y$. In the rest of the proof we assume that $\hat{N}_y$ is co-inner without loss of generality. By Proposition 1, there exists a solution to the stabilization problem i.e. there exists an $\hat{S} \in z^1 H^\infty$ such that $\hat{G}_e \in L^\infty$ if all conditions of Proposition 1 are satisfied.

As $R := \begin{bmatrix} \hat{H}_o^{-1} & I - \hat{H}_o \hat{H}_o \end{bmatrix}$ satisfies $R^* R = I$, we have

$$\| \hat{G}_v - \hat{H} \hat{S} \hat{G}_y \|_{L^2} = \| R \hat{G}_v - \hat{H} \hat{H}_o \hat{S} \hat{G}_y \|_{L^2}$$

$$= \left\| \begin{bmatrix} \hat{H}_o^{-1} \hat{G}_v - \hat{H}_o \hat{S} \hat{G}_y \\ \hat{H}_o \hat{G}_v \end{bmatrix} \right\|_{L^2}$$

Therefore, $\| \hat{G}_v - \hat{H} \hat{S} \hat{G}_y \|_{L^2}$ is finite iff $\| \hat{H}_o^{-1} \hat{G}_v - \hat{H}_o \hat{S} \hat{G}_y \|_{L^2}$ and $\| \hat{H}_o \hat{G}_v \|_{L^2}$ are finite. Since $\hat{H}_o^{-1} \hat{G}_v - \hat{H}_o \hat{S} \hat{G}_y \in L^\infty$ (by (62)) and is a sampler, it belong to $L^2$ also [13, proposition 5.3]. Therefore $\hat{G}_e \in L^2$ iff $\hat{H}_o \hat{G}_v \in L^2$.

Since $\hat{S} \in z^1 H^\infty$ and all conditions of Proposition 1 are satisfied, by Proposition 2 we can parameterize $\hat{S}$ and $\hat{G}_e \in L^\infty$ in term of $\hat{S}_\alpha \in z^1 H^\infty$ as given in (10) and (11) respectively. Define

$$\hat{S}_\alpha := \arg \inf_{\hat{S}_\alpha \in z^1 H^\infty} \| \hat{G}_v + \hat{H}_o \hat{M}_o \hat{G}_y - \hat{H}_o \hat{S}_\alpha \hat{N}_y \|.$$
Since \( \tilde{G}_o \in L^2 \), we use projections to say that \( \hat{S}_o \) must satisfy the following:
\[
\left\langle \tilde{G}_e, \hat{H}_i \hat{S}_o \hat{N}_y \right\rangle = \left\langle \tilde{G}_v + \hat{H}_i \hat{M}_h \tilde{G}_v - \hat{H}_i \hat{S}_o \hat{N}_y, \hat{H}_i \hat{S}_o \hat{N}_y \right\rangle = 0
\]
for all \( \hat{S}_o \in \mathbb{C}^4. \) This can be achieved if we take \( \hat{S}_o = \hat{S}_{o, \text{opt}}. \) In particular
\[
\left\langle \tilde{G}_{e, \text{opt}}, \hat{H}_i \hat{S}_{o, \text{opt}} \hat{N}_y \right\rangle = 0,
\]
Therefore (by Pythagoras theorem),
\[
\| \tilde{G}_v + \hat{H}_i \hat{M}_h \tilde{G}_v - \hat{H}_i \hat{S}_{o, \text{opt}} \hat{N}_y \|_2^2 = \| \hat{H}_i \hat{S}_{o, \text{opt}} \hat{N}_y \|_2^2
\]
\[
\| \hat{H}_i \|_2^2 \| \hat{N}_y \|_2^2.
\]
Since \( \hat{H}_i \) is inner and \( \hat{N}_y \) is co-inner, we have (14).
Since \( \hat{H}_i^\dagger \hat{N}_y \hat{N}_y^\dagger - \hat{V} \in L^\infty \cap L^2 \), the \( \hat{S}_{o, \text{opt}} \) of (13) is in \( L^\infty \cap L^2 \) as well. Hence \( \hat{S}_{o, \text{opt}} \in L^\infty \cap L^2 \).

11.2 Proofs of the results in Section 7

**Proof (Lemma 3):** Note that if \( Ye^{Ah} \) is Schur then \( (zI - Ye^{Ah})^{-1} \) exists for all \( z \in \mathbb{C} \setminus \mathbb{D} \). Using this, the proof follows from continuity (hence bounded on a compact set) of \( e^{At} \), \( B \) and \( C \). \( \square \)

**Proof (Lemma 4):** The proof follows from the fact that constant multiplicative operators mapping \( L^2[0,h) \) to \( L^2[0,h) \) are not compact (hence not Hilbert-Schmidt). \( \square \)

**Proof (Lemma 5):** Using (19), we have that
\[
\tilde{G} = \tilde{X} - \tilde{Y}
\]
where
\[
\tilde{X}(z) \tilde{u}(z) := C(\tau) \int_0^\tau e^{A(\tau - \sigma)} B(\sigma) \tilde{u}(\sigma) \, d\sigma, \quad \tau \in [0,h)
\]
\[
\tilde{Y}(z) \tilde{u}(z) := C(\tau)e^{At}(\Omega(z) + \Theta(z)e^{Ah})^{-1}\Theta(z) \int_0^h e^{A(h - \sigma)} B(\sigma) \tilde{u}(\sigma) \, d\sigma.
\]
Since \( e^{At}, B \) and \( C \) are continuous on \( [0,h] \) (hence bounded on compact set \( [0,h] \)) and \( \tilde{X} \) is causal, \( \tilde{X} \in H^\infty \cap H^2. \)
Since \( \tilde{Y}(z) \) is a hybrid signal processor (i.e. a cascade of sampler, discrete system and a hold), we have that rank \( \tilde{Y}(e^{Ah}) \) is uniformly bounded for all \( \theta \in [-\pi, \pi]. \) Therefore if \( \tilde{Y} \in H^\infty, \) it is in \( H^2 [13, \text{proposition 5.3}]. \) \( \square \)

**Proof (Lemma 6):** Since \( \tilde{B} \in C^2([0,h], \mathbb{C}^{n \times m}) \), we have that the integral \( \int_0^h B(\sigma)B(\sigma)^* d\sigma \) is well-defined in \( \mathbb{C}^{n \times n}. \)
Therefore,
\[
\int_0^h e^{A(h - \sigma)} B(\sigma)B(\sigma)^* e^{A^*(h - \sigma)} \, d\sigma
\]
is a well-defined matrix (note that \( e^{A(h - \sigma)} \) is a bounded function on a compact set). Similarly, it can be proven that
\[
\int_0^h e^{A^*(\tau)} C^*(\tau)C(\tau) e^{A^t} \, d\tau
\]
is a well-defined matrix. Since both of the integral above are non-negative, there exist matrices \( \tilde{D} \) and \( \tilde{C} \) of the form given in (27).
Using (19), we have
\[
\tilde{G} = \tilde{D} + \tilde{Y}
\]
where

\[ \tilde{Y}(z) = C(\tau)e^{A\tau}(zI - Ye^{Ah})^{-1}T \int_0^h e^{A(h-\sigma)}B(\sigma)\hat{u}(\sigma)\,d\sigma. \]

Since \( e^{A\tau} \), \( B \) and \( C \) are continuous on \([0,h]\) (hence bounded on compact set \([0,h]\)), \( \tilde{D} \in H^2 \). By definition, \( \|\tilde{D}\|_{HS}^2 = \frac{1}{2}\|\tilde{D}\|_{H^2}^2 \).

Since \( \tilde{G}(z) \) is causal, \( \tilde{Y}(z) \) is also causal. The formal series of causal \( \tilde{Y}(z) \) has the constant term

\[ \lim_{z \to \infty} \tilde{Y}(z) = 0. \]

Therefore, if \( \tilde{G} \in H^2 \) then by orthogonality between \( \tilde{D} \) and \( \tilde{Y} \) in the space \( H^2 \), we have

\[ \|\tilde{G}\|^2_{H^2} = \frac{1}{2}\|\tilde{D}\|^2_{HS} + \|\tilde{Y}\|^2_{H^2}. \]

The integral form of \( \|\tilde{D}\|^2_{HS} \) can be obtained by using \( \|\tilde{D}\|^2_{HS} = \text{tr}\,\tilde{D}\tilde{D}^* \).

To calculate \( H^2 \) norm of \( \tilde{Y} \), define a zero order hold \( C \) as

\[ \tilde{y}(z) = \tilde{C}(z)x(z): \tilde{y}(z;\tau) = C(\tau)e^{A\tau}x(z) \]  \hspace{1cm} (63)

Clearly, \( \tilde{C}^*\tilde{C} \) is a static discrete system given by

\[ \tilde{C}^*\tilde{C} = \tilde{C}^*\tilde{C} \]

Define a zero order hold \( \tilde{V}_L := \tilde{C}\tilde{C}^+ \) where \( \tilde{C}^+ \) is the pseudo-inverse of the matrix \( C \). Note that \( \tilde{V}_L^*\tilde{V}_L = \tilde{C}\tilde{C}^+ \) i.e it is an orthogonal projection onto \( \text{Im}\,\tilde{C} \). Clearly, \( \tilde{V}_L \in H^2 \) (follows from continuity of \( C(\tau) \) and \( e^{A\tau} \)). Also, we have \( \tilde{V}_L\tilde{V}_L^*\tilde{C} = \tilde{C} \).

Define a sampler \( \tilde{B} \) as

\[ \tilde{y}(z) = \tilde{B}(z)x(z): \tilde{y}(z) = \int_0^h e^{A(h-\sigma)}B(\sigma)x(z;\sigma)d\sigma \]  \hspace{1cm} (64)

Also, define a sampler \( \tilde{V}_R := \tilde{B}^+\tilde{B} \) where \( \tilde{B}^+ \) is the pseudo-inverse of the matrix \( \tilde{B} \). Note that, \( \tilde{V}_R\tilde{V}_R = \tilde{B}^+\tilde{B} \) i.e. it is an orthogonal projection onto \( (\text{Ker}\,\tilde{B})^\perp \). Clearly, \( \tilde{V}_R \in H^2 \). Also, we have \( \tilde{B}\tilde{V}_R^*\tilde{V}_R = \tilde{B} \).

Therefore, using \( \tilde{V}_L\tilde{V}_L^*\tilde{C} = \tilde{C} \) and \( \tilde{B}\tilde{V}_R^*\tilde{V}_R = \tilde{B} \), we have

\[ \tilde{Y} = \tilde{V}_L\tilde{V}_R \]

where \( \tilde{Y} := \tilde{V}_L^*\tilde{Y}\tilde{V}_R^* \) is discrete system given in state-space as

\[ \tilde{Y} = \begin{pmatrix} e^{Ah} & \tilde{B} \\ \tilde{C} & \tilde{V}_R^* \end{pmatrix} \]

Here, we used

\[ (\tilde{C}^+)^*\tilde{C}^*\tilde{C} = (\tilde{C}^+)^*\tilde{C}^*\tilde{C} = (\tilde{C}\tilde{C}^+)\tilde{C}^+ \tilde{C}^* \tilde{C} = \tilde{C}\tilde{C}^+ \tilde{C} = \tilde{C} \]

\[ \tilde{B}\tilde{B}^* = \tilde{B}\tilde{B}^* = \tilde{B}\tilde{B}^* = \tilde{B}\tilde{B}^* = \tilde{B}\tilde{B}^* = \tilde{B} \]

which follows from the fact that \( \tilde{C}\tilde{C}^+ \) and \( \tilde{B}\tilde{B}^* \) are orthogonal projections.

Since \( \tilde{V}_L^*\tilde{V}_L = \tilde{C}\tilde{C}^+ \), \( \tilde{V}_R\tilde{V}_R^* = \tilde{B}^+\tilde{B} \) and \( \tilde{V}_L^*, \tilde{V}_R^* \in H^2 \), we have

\[ \|\tilde{Y}\|^2_{H^2} = \|\tilde{V}_L\tilde{V}_R\|^2_{H^2} = \|\tilde{Y}\|^2_{H^2}. \]

\[ \Box \]

Proof (Lemma 7): The proof is similar to the proof Lemma 6. We also used

\[ \|\tilde{C}(zI - Ye^{Ah})^{-1}B(r)\|^2_{H^2} = \|\tilde{C}(zI - Ye^{Ah})^{-1}(zI)B(r)\|^2_{H^2} \]

in the proof. \( \Box \)
11.3 Proofs of the results in Section 8.

Proof (Lemma 9): The coprime factorization is quite standard (see [31, chap. 5]). $\tilde{M}_\nu, \tilde{M}_c, \tilde{N}_\nu$ and $\tilde{N}_c$ belong to $H^\infty$ because $e^{(A+LC_y)\theta}$ is Schur (see Lemma 3).

Proof (Lemma 10): Note that $N_\nu$ is an LCTI system with state-space

$$
N_\nu = \begin{bmatrix}
A + LC_y & B + LD_y \\
Z_y C_y & Z_y D_y
\end{bmatrix},
$$

where $A + LC_y$ is Hurwitz. Now, if we have $\tilde{N}_\nu(e^{j\theta})\tilde{N}_\nu^{-1}(e^{j\theta}) > 0\forall \theta \in [-\pi, \pi]$ then $N_\nu N_\nu^{-1} > 0$ and $N_\nu(j\omega)N_\nu(j\omega)^{-1} > 0$ for all $\omega \in \mathbb{R}$ including $\infty$. Here, $N_\nu$ is the system $\tilde{N}_\nu$ in the time domain and $N_\nu$ is the system $\tilde{N}_\nu$ in the (classic) frequency domain. As $Z_y$ is invertible and $A + LC_y$ is Hurwitz, $N_\nu(j\omega)N_\nu(j\omega)^{-1} > 0$ for all $\omega \in \mathbb{R}$ including $\infty$ iff $D_y$ has full row rank and

$$
\begin{bmatrix}
A + LC_y - j\omega I & B + LD_y \\
Z_y C_y & Z_y D_y
\end{bmatrix} = \begin{bmatrix} I & L \\
0 & Z_y \end{bmatrix} \begin{bmatrix} A - j\omega I & B \\
C_y & D_y \end{bmatrix}
$$

has full row rank for all $\omega \in \mathbb{R}$.

Proof (Lemma 11): Follows from Lemma 10 and [32, Theorem 13.35].

Proof (Lemma 12): The proof is quite standard.

Proof (Proposition 4): Define $\tilde{H}_{cs}(z) := \tilde{C}_H \tilde{H}_s(z)$. Note that

$$
\hat{H}^{-1}(z) \hat{H}(z) = \hat{H}_s(z) \hat{C}_H \hat{H}_s(z) = \hat{H}_{cs}(z) \hat{H}_{cs}(z).
$$

As $Ee^{A_nh}$ is Schur (Assumption $A_8$), $\hat{H}_{cs}(e^{j\theta})\hat{H}_{cs}(e^{j\theta}) > 0\forall \theta \in [-\pi, \pi]$ is equivalent to say that the matrix

$$
R(e^{j\theta}) := \begin{bmatrix}
Ee^{A_nh} - e^{j\theta} I & Ee^{A_nh}B_H \\
\hat{C}_H & \hat{C}_H B_H
\end{bmatrix}
$$

has full column rank for every $\theta \in [-\pi, \pi]$. Now, $R(e^{j\theta})$ can be written as

$$
R(e^{j\theta}) = \begin{bmatrix}
Ee^{A_nh} - e^{j\theta} I & Ee^{A_nh}B_H \\
\hat{C}_H & \hat{C}_H B_H
\end{bmatrix} \begin{bmatrix} I & B_H \\
0 & e^{j\theta} I \end{bmatrix}.
$$

Since $\begin{bmatrix} I & B_H \\
0 & e^{j\theta} I \end{bmatrix}$ is invertible for all $\theta \in [-\pi, \pi]$, we have that

$$
\text{rank } R(e^{j\theta}) = \text{rank } \begin{bmatrix}
Ee^{A_nh} - e^{j\theta} I & Ee^{A_nh}B_H \\
\hat{C}_H & \hat{C}_H B_H
\end{bmatrix}.
$$

for all $\theta \in [-\pi, \pi]$. Now, the proof of equivalence of condition 1 and 2 is well known and the proof of equivalence of condition 2 and 3 is essentially given in [15, Theorem 4.1]. Note that if $\hat{H} = \hat{H}_{0} \hat{H}_o$, then a spectral factor $W(z)$ of $\hat{H}(z)\hat{H}(z)$ is $\hat{H}_o$ and if $W(z)$ is a spectral factor of $\hat{H}(z)\hat{H}(z)$ then $\hat{H}_o(z) = W(z)$ and $\hat{H}_o(z) = H(z)W^{-1}(z)$. Now, we prove the equivalence of condition 2 and 4. Note that

$$
\hat{H}_{cs}(z) = \hat{C}_H B_H + \hat{C}_H (zI - Ee^{A_nh})^{-1}Ee^{A_nh}B_H
= \hat{C}_H (zI - Ee^{A_nh})^{-1}(zI - Ee^{A_nh} + Ee^{A_nh})B_H
= z \hat{C}_H (zI - Ee^{A_nh})^{-1}B_H.
$$

This implies,

$$
\hat{H}_{cs}(z)\hat{H}_{cs}(z) = B_H \left(\frac{1}{z} - (Ee^{A_nh})^{-1}\right) \hat{C}_H \hat{H}_s(zI - Ee^{A_nh})^{-1}B_H.
$$

Using the above equation and [15, Theorem 4.1], the existence of the spectral factorization of $\hat{H}(z)\hat{H}(z) = \hat{H}_{cs}(z)\hat{H}_{cs}(z)$ is equivalent to 4.
Proof (Lemma 13) : Since \((C_H, A_H)\) is observable, we have \(\hat{C}_H^*\hat{C}_H > 0\). This implies \(\hat{C}_H\) have full column rank. As \(B_H\) and \(\hat{C}_H\) has full column rank, condition 3 of Proposition 4 is satisfied. Now, the results follows from Proposition 4. □

Proof (Lemma 14) : The state equation of the Hold \(\hat{H}\) can be written as

\[
\dot{x}(\tau) = A_H \dot{x}(\tau) + B_H J_0+ \dot{u}, \quad z\dot{x}(0) = E\dot{x}(h^-), \quad \tau \in [0,h)
\]

\[
\dot{y}(\tau) = C_H \dot{x}(\tau)
\]

Using the standard trick of state feedback for constructing coprime factors, we define

\[
\bar{v} := \bar{u} - F\xi(0^-) = \bar{u} - \frac{1}{z} F J_0 \bar{x}(\tau).
\]

Therefore,

\[
\dot{x}(\tau) = A_H \dot{x}(\tau) + B_H J_0+ (\bar{v} + F\xi(0^-)), \quad z\dot{x}(0) = E\dot{x}(h^-), \quad \tau \in [0,h)
\]

\[
\dot{y}(\tau) = C_H \dot{x}(\tau)
\]

Now using [16, Proposition A.2], we have

\[
\dot{x}_1(\tau) = A_H \dot{x}_1(\tau) + B_H J_0+ \bar{v}, \quad z(\dot{x}_1(0) - B_H F\dot{x}_1(0^-)) = E\dot{x}(h^-)
\]

Now, consider the boundary condition

\[
z(\dot{x}_1(0) - B_H F\dot{x}_1(0^-)) = E\dot{x}_1(h^-)
\]

\[
\iff z(\dot{x}_1(0) - \frac{1}{z} B_H F\dot{x}_1(h^-)) = E\dot{x}_1(h^-)
\]

\[
\iff \dot{x}_1(0) = E\dot{x}_1(h^-) + B_H F\dot{x}_1(h^-) = (E + B_H F)\dot{x}_1(h^-)
\]

Now, \(\bar{u} = \hat{M}_H \bar{v}, \bar{y} = \hat{N}_H \bar{v}\). Since \(F\) is such that \((E + B_H F)\) is Schur, therefore \(\hat{M}_H, \hat{N}_H \in H^\infty\) (see Lemma 3). The mapping \(\hat{M}_H^{-1}\) from \(\bar{u}\) to \(\bar{v}\) is given by

\[
\hat{M}_H^{-1}(z) = \begin{bmatrix}
A_H & B_H J_0+ \\
-\frac{1}{z} J_0 F & I
\end{bmatrix} \begin{bmatrix}
E \end{bmatrix}
\]

Since \(E\hat{A} \hat{A} H\) is Schur (see Assumption \(\mathcal{A}_5\)), discrete system \(\hat{M}_H^{-1}\) is in \(H^\infty\) (as \(\hat{M}_H^{-1}(z)\) is analytic and bounded in \(\mathbb{C} \setminus \mathbb{D}\)). Now, we have

\[
\hat{M}_H^{-1} \hat{M}_H + 0 \hat{N}_H = I
\]

Therefore, \(\hat{M}_H\) and \(\hat{N}_H\) are right coprime. □

Proof (Lemma 15) : For some invertible complex matrix \(Z\), our aim here is to make \(\hat{N}_H Z\) an inner matrix i.e \((\hat{N}_H Z)^\sim \hat{N}_H Z = I\). The conjugate of the \(\hat{N}_H\) is the sampler \(\hat{N}_H^\sim\) with

\[
\hat{N}_H^\sim(z) = J_0^+ \begin{bmatrix}
-A_H & C_H^* \\
-B_H & 0
\end{bmatrix} \begin{bmatrix}
E + B_H F & -I
\end{bmatrix}.
\]

To find the \(Z\) such that \((\hat{N}_H Z)^\sim \hat{N}_H Z = I\), we first consider

\[
\hat{N}_H(z) \hat{N}_H(z) = J_0^+ \begin{bmatrix}
-A_H & C_H^* C_H \\ -B_H & 0
\end{bmatrix} \begin{bmatrix}
E + B_H F & 0 & 0 \\ 0 & B_H & 0
\end{bmatrix} J_0^+ \begin{bmatrix}
E \end{bmatrix}
\]

where \(\Omega_0 := \begin{bmatrix}
(E + B_H F)^* \\
0
\end{bmatrix}\) and \(\mathcal{Y}_0 := - \begin{bmatrix}
I & 0 \\ 0 & E + B_H F
\end{bmatrix}\).
Applying a time varying state transform $T(t) = \begin{bmatrix} I & Q(t) \\ 0 & I \end{bmatrix}$ where $Q(t)$ satisfy the differential Lyapunov equation

$$
\dot{Q}(t) = -A_H^*Q(t) - Q(t)A_H - C_H^*C_H,
$$

we have

$$
\dot{\tilde{N}}_H(z) = \tilde{J}_0^+ \begin{bmatrix} -A_H^* & 0 & Q(t)B_H \\ 0 & A_H & B_H \\ -B_H^* & B_HQ(t) & 0 \end{bmatrix} \begin{bmatrix} z\Omega_0^{-1}(0) & T_0^{-1}(h) \end{bmatrix} J_0^+
$$

Now, the boundary condition of $\tilde{N}_H(z)$ is given by

$$
\begin{bmatrix} (E + B_HF)^* & 0 & I - Q_0 \\ 0 & I & 0 \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & E + B_HF \end{bmatrix} \begin{bmatrix} I - Q_h \\ 0 \end{bmatrix}
$$

which can be rearranged as

$$
\begin{bmatrix} (E + B_HF)^* & 0 & I - Q_0 \\ 0 & I & 0 \end{bmatrix} - \begin{bmatrix} I - Q_h \\ 0 \end{bmatrix} (E + B_HF).
$$

Here $Q_0 := Q(0)$ and $Q_h := Q(h)$. To decouple the boundary condition (meaning block diagonal here), pre-multiply both sides above by

$$
S := \begin{bmatrix} I & (E + B_HF)^*Q_0 \\ 0 & I \end{bmatrix}.
$$

Pre-multiplication with $S$ does not change the system (see Lemma 2), therefore we have the boundary condition

$$
\begin{bmatrix} (E + B_HF)^* & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} I - Q_h + (E + B_HF)^*Q_0(E + B_HF) \\ 0 \end{bmatrix} E + B_HF
$$

The condition

$$(E + B_HF)^*Q_0(E + B_HF) = Q_h = 0
$$

will lead to decoupled STPBC.

The solution of differential Lyapunov equation (66) with initial condition $Q(0) = Q_0$ is given by

$$
Q(t) = R_1(t)Q_0R_1^*(t) - R_3(t)
$$

where $R_1(t) := e^{-A_Ht}$, and $R_3(t) := \int_0^t e^{-A_Hs}C_HC_H e^{-A_Hs} ds$ [7, chapter 8]. Now using decoupling condition (67), we have

$$
Q(h) = R_1(h)Q_0R_1^*(h) - R_3(h) = (E + B_HF)^*Q_0(E + B_HF).
$$

The above equation can be written in $Q_0$ alone as

$$
Q_0 = R_1(h)^{-1} ((E + B_HF)^*Q_0(E + B_HF) + R_3(h)) R_1(h)^{-*} = R_1(h)^{-1} (E + B_HF)^*Q_0(E + B_HF) R_1(h)^{-*} + C_H^*C_H
$$

As $(A_H, C_H)$ is assumed observable (Assumption $A_7$), $C_H^*C_H > 0$ [32, theorem 3.3]. Also, $(E + B_HF)R_1(h)^{-*}$ is Schur because $F$ is assumed to be chosen that way. Therefore, there exists a unique solution $Q_0 > 0$ of discrete Lyapunov equation (68).

This value of $Q_0$ renders

$$
\tilde{N}_H(z) = \tilde{J}_0^+ \begin{bmatrix} -A_H^* & 0 & Q(t)B_H \\ 0 & A_H & B_H \\ -B_H^* & B_HQ(t) & 0 \end{bmatrix} \begin{bmatrix} z\Omega_0^{-1} & T_0 \end{bmatrix} J_0^+.
$$
with decoupled boundary condition. Next, we look for a matrix $F$ such that $\hat{N}_H(z)\hat{N}_H(z)$ is a static (and invertible also) discrete system. Using [17, lemma 4], we have that

$$\hat{N}_H(z)\hat{N}_H(z) = -B_H^t(z(E + B_H F)^* - e^{-A_H^h})^{-1}z(E + B_H F)^*Q_0B_H$$

$$+ B_H^t Q_0(zI - (E + B_H F)e^{A_H^h})^{-1}zB_H.$$ Since,

$$B_H^t Q_0(zI - (E + B_H F)e^{A_H^h})^{-1}zB_H$$

$$= B_H^t Q_0(I + z^{-1}(E + B_H F)e^{A_H^h}(I - z^{-1}(E + B_H F)e^{A_H^h})^{-1})B_H$$

$$= B_H^t Q_0B_H + B_H^t Q_0(E + B_H F)e^{A_H^h}(zI - (E + B_H F)e^{A_H^h})^{-1}B_H$$

This implies that if we choose $F$ such that $B_H^t Q_0(E + B_H F) = 0$, then

$$\hat{N}_H(z)\hat{N}_H(z) = B_H^t Q_0B_H$$

which is a static discrete system. Since $\bar{z}$ where we used $z$ is the pseudo-inverse of the matrix $\bar{H}$, we have that $\bar{z}$ is invertible, hence

$$F = -(B_H^t Q_0B_H)^{-1}B_H^t Q_0E$$

and $\hat{N}_H(z)(B_H^t Q_0B_H)^{-1}$ is the desired inner factor.

To prove that $F = -(B_H^t Q_0B_H)^{-1}B_H^t Q_0E$ makes $(E + B_H F)e^{A_H^h}$ Schur, we substitute this value of $F$ in (68). Hence,

$$E = (E + B_H F)^*Q_0(E + B_H F)$$

$$= E^*Q_0E - E^*Q_0B_zQ_0E$$

$$= E^*Q_0E - E^*Q_0B_zQ_0E$$

where we used $B_z := B_H(B_H^t Q_0B_H)^{-1}B_H^t$ and that $B_zQ_0B_z = B_z$. Therefore,

$$Q_0 = R_1(h)^{-1}E^*(Q_0 - Q_0B_H(B_H^t Q_0B_H)^{-1}B_H^t Q_0)ER_1(h)^{-1} + \hat{C}_H^t \hat{C}_H$$

By Lemma 13, Assumption $A_2$ is satisfied as assumptions $A_5 - A_7$ are satisfied. Now, if Assumption $A_2$ is satisfied then there exists a unique solution $Q_0$ such that $(E + B_H F)e^{A_H^h}$ is Schur matrix (see Proposition 4). □

Proof (Lemma 16) : Assume $C_H$ is a matrix which satisfy (39) and let the discrete system $\hat{H}_a$ be as in (38). Let $\hat{V}_H$ be a zero order hold defined as

$$\bar{y}(z) = \hat{V}_H(z)\bar{x}(z) : \bar{y}(z; \tau) = \int_0^h C_H e^{A_H^h} \bar{C}_H^+ \bar{x}(z)d\tau, \quad \tau \in [0, h)$$

where $\bar{C}_H^+$ is the pseudo-inverse of the matrix $\bar{C}_H$. Note that $\hat{V}_H\hat{V}_H^t \bar{C}_H = \bar{C}_H$ where $\bar{C}_H \in \mathbb{H}^\infty$ is a lifted zero order hold whose hold function in lifted $z$-domain is given by $C_H e^{A_H^h}$ (see [23, lemma 5.4.11]). Using [23, lemma 5.4.11], we can write $\hat{H} = \hat{V}_H \hat{H}_{ics}$ and $\hat{H}_a = \hat{V}_H \hat{H}_{ics}$ where

$$\hat{H}_{ics} := \begin{pmatrix}
(E + B_H F)e^{A_H^h} & (E + B_H F)e^{A_H^h}B_H \\
\bar{C}_H B_H Z
\end{pmatrix}.$$

Since $\hat{H}_a$ is bicausal and bistable, it will not have zeros in the region $|z| \geq 1$. Therefore, a zero of $\hat{H}_{ics}$ in the region $|z| \geq 1$ is a zero of $\hat{H}_{ics}$. Now $\hat{H}_{ics}^t \hat{V}_H^t = \hat{V}_H \hat{V}_H^t \hat{V}_H \hat{H}_{ics}$. As $\hat{V}_H^t$ is a sampler with no poles, the poles of $\hat{H}_{ics}^t$ are the poles of $\hat{H}_{ics}$. Also, since $\hat{H}_{ics}^t \hat{H}_{ics} = I$, the poles of $\hat{H}_{ics}$ are the zero of $\hat{H}_{ics}$. Therefore, Assumption $A_8$ can be restated as “no pole of $\hat{H}_{ics}$ in the region $|z| \geq 1$ is a pole of $\hat{C}_H$ (or $\hat{C}_H^+$).” □

Proof (Lemma 17) : The solution of Sylvester differential equation (46) with boundary condition $P(0) = P_0$ is given by

$$P(\tau) = R_1(\tau)P_0 R_2(\tau) - R_3(\tau)$$

where $R_1(\tau) := e^{-A_H^h \tau}$ and $R_2(\tau) := e^{-A_H \tau}$ [7, chapter 8]. Using above and the boundary condition $(E + B_H F)^*P(0) = -P(h)$, we have

$$P(h) = R_1(h)P_0 R_2(h) - R_3(h) = -(E + B_H F)^*P_0$$
this implies
\[ P_0 = -R_1(h)^{-1}(E + B_H F)^* P_0 R_2(h)^{-1} + R_1(h)^{-1} R_2(h) R_2(h)^{-1} \]

The unique solution of the above discrete Sylvester equation exists by Lemma 16. □

**Proof (Lemma 18):**

Since the \( P(\tau) \) exists (see Lemma 17) and continuous on \([0, h]\) (hence, bounded), and \((E + B_H F)e^{A_{H}^h}\) is Schur, it follows from Lemma 3, \( \dot{V}^* \in H^\infty \). Hence \( \dot{V} \in L^\infty \).

Now, we have the boundary condition
\[ \dot{V}(z) \dot{M}(z) = \mathcal{J}_0^+ \begin{bmatrix} -A_H^* & C_H C_v & 0 \\ 0 & A + LC_y & L \\ -(B_H Z)^* & 0 & 0 \end{bmatrix} \begin{bmatrix} z \Omega_p - I \end{bmatrix} \]

and
\[ V(z) \dot{M}(z) = \mathcal{J}_0^+ \begin{bmatrix} -A_H^* & -P(\tau) LC_y & -P(\tau) L \\ 0 & A + LC_y & L \\ -(B_H Z)^* & 0 & 0 \end{bmatrix} \begin{bmatrix} z \Omega_p - I \end{bmatrix} \]

where \( \Omega_p := \begin{bmatrix} (E + B_H F)^* & 0 \\ 0 & I \end{bmatrix} \). Hence,
\[ \dot{M}_h(z) = \mathcal{J}_0^+ \begin{bmatrix} -A_H^* & C_H C_v - P(\tau) LC_y & -P(\tau) L \\ 0 & A + LC_y & L \\ -(B_H Z)^* & 0 & 0 \end{bmatrix} \begin{bmatrix} z \Omega_p - I \end{bmatrix} \]

Using a time-varying state transform \( T(\tau) = \begin{bmatrix} I & P_1(\tau) \\ 0 & I \end{bmatrix} \), where \( P_1(\tau) \) satisfy
\[ \dot{P}_1(\tau) + A_H^* P_1(\tau) + P_1(\tau) (A + LC_y) + C_H^* C_v - P(\tau) LC_y = 0, \]

Now, we have the boundary condition
\[ \begin{bmatrix} z \begin{bmatrix} (E + B_H F)^* & -(E + B_H F)^* P_1(0) \end{bmatrix} - \begin{bmatrix} I & -P_1(0) \end{bmatrix} \end{bmatrix} \]

Decoupling the boundary condition by premultiplying with matrix \( \begin{bmatrix} I & P_1(0) \\ 0 & I \end{bmatrix} \), we have
\[ \begin{bmatrix} z \begin{bmatrix} (E + B_H F)^* & -(E + B_H F)^* P_1(0) - P_1(0) \end{bmatrix} - \begin{bmatrix} I & 0 \end{bmatrix} \end{bmatrix} \]

Therefore \((E + B_H F)^* P_1(0) = -P_1(0)\), would guarantee de-coupled states.

If we take \( P_1(\tau) = P(\tau) \), we have
\[ \dot{P}(\tau) + A_H^* P(\tau) + P(\tau) A + C_H^* C_v = 0, \]

with boundary condition \((E + B_H F)^* P(0) = -P(h)\), which we know is true by (46). Hence
\[ \dot{M}_h(z) = \mathcal{J}_0^+ \begin{bmatrix} -A_H^* & 0 & 0 \\ 0 & A + LC_y & L \\ -(B_H Z)^* & -(B_H Z)^* P(\tau) & 0 \end{bmatrix} \begin{bmatrix} z \Omega_p - I \end{bmatrix} \]
\[ = \mathcal{J}_0^+ \begin{bmatrix} A + LC_y & L \\ -(B_H Z)^* & 0 \end{bmatrix} \begin{bmatrix} z I - I \end{bmatrix} \]

where we used \( T(\tau) \begin{bmatrix} -P(\tau) L \\ L \end{bmatrix} = \begin{bmatrix} 0 \\ L \end{bmatrix} \). Since \( A + LC_y \) is Hurwitz, \( e^{(A+LC_y)h} \) is Schur. Therefore, it follows from Lemma 3 that \( \dot{M}_h \in H^\infty \). □
Proof (Corollary 2): Using Lemma 12, we have
\[ \dot{H}_\nu(z) \dot{N}_\nu(z) \dot{N}_\nu^* (z) - \dot{V}(z) \]
\[ = J_{0^+}^* \begin{bmatrix} A_p & 0 \\ C_p & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \Omega_p - I \end{bmatrix} - \dot{V}(z) \]
\[ = J_{0^+}^* \begin{bmatrix} -A_H^* & 0 & 0 \\ 0 & -A_H^* & -C_H C_v X \\ 0 & 0 & -A_L^* \end{bmatrix} \begin{bmatrix} -P(\tau)LZ_y^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} (Z_y C_y)^* \\ 0 \end{bmatrix} \begin{bmatrix} (B H Z)^* - (B H Z)^* \\ 0 \end{bmatrix} \begin{bmatrix} \Omega_1 - I \end{bmatrix} \]
where
\[ \Omega_1 := \begin{bmatrix} (E + B_H F)^* & 0 & 0 \\ 0 & (E + B_H F)^* & 0 \\ 0 & 0 & I \end{bmatrix} \]

Applying a transform \( T = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \), we have
\[ \dot{H}_\nu(z) \dot{N}_\nu(z) \dot{N}_\nu^* (z) - \dot{V}(z) \]
\[ = J_{0^+}^* \begin{bmatrix} -A_H^* & 0 & 0 \\ 0 & -A_H^* & -C_H C_v X \\ 0 & 0 & -A_L^* \end{bmatrix} \begin{bmatrix} -P(\tau)LZ_y^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} (Z_y C_y)^* \\ 0 \end{bmatrix} \begin{bmatrix} (B H Z)^* - (B H Z)^* \\ 0 \end{bmatrix} \begin{bmatrix} \Omega_2 - T^{-1} \end{bmatrix} \]
where
\[ \Omega_2 := \begin{bmatrix} (E + B_H F)^* & 0 & 0 \\ 0 & (E + B_H F)^* & 0 \\ 0 & 0 & I \end{bmatrix} \]

Multiplying the boundary condition by an invertible matrix does not change system, therefore we multiply with \( S := T \), therefore we have
\[ \dot{H}_\nu(z) \dot{N}_\nu(z) \dot{N}_\nu^* (z) - \dot{V}(z) \]
\[ = J_{0^+}^* \begin{bmatrix} A_p & 0 \\ C_p & 0 \end{bmatrix} \begin{bmatrix} B_p \\ 0 \end{bmatrix} \begin{bmatrix} \Omega_p - I \end{bmatrix} \]
\[ \square \]

Proof (Theorem 2): To obtain \( \dot{S}_{\alpha, \text{opt}} \) consider the STPBC realization of \( \dot{Y} := \dot{H}_\nu^{-1} \dot{N}_\nu \dot{N}_\nu^* - \dot{V} \) given in Corollary 2 i.e.
\[ \dot{Y}(z) = J_{0^+}^* \begin{bmatrix} A_p \\ C_p \end{bmatrix} \begin{bmatrix} B_p \\ 0 \end{bmatrix} \begin{bmatrix} \Omega_p - I \end{bmatrix} \]
where all eigenvalues of \( e^{-A_H \Omega_p} \) are in the region \(|z| > 1\). Define a sampler
\[ \dot{T}(z) \dot{u}(z) := \int_0^h e^{A_p(h-\sigma)}B \dot{u}(z; \sigma)d\sigma. \]
To calculate $\dot{S}_{a, \text{opt}} = \text{proj}_{z^1H^2} \dot{Y}$ consider

$$
\dot{Y}(z) = C_p(z\Omega_p - e^{Ah})^{-1} T(z) \dot{u}(z) \\
= C_p(ze^{-Ah}\Omega_p - I)^{-1} T \dot{u}(z) \\
= -C_p \left( I + ze^{-Ah}\Omega_p + \cdots + (ze^{-Ah}\Omega_p)^j \right) T \dot{u}(z) \\
- C_p(ze^{-Ah}\Omega_p)^{j+1} \left( I + (ze^{-Ah}\Omega_p^{j+1}) \right) T \dot{u}(z)
$$

Now,

$$
\text{proj}_{z^1H^2} \dot{Y}(z) = \dot{Y}(z) + C_p(ze^{-Ah}\Omega_p)^{j+1} (\dot{z} \Omega_p - e^{Ah})^{-1} T \dot{u}(z)
$$

The rest of the proof follows from Lemma 11, Lemma 15, Theorem 1, and Corollary 1. □

**Proof (Lemma 20):**

The integral equalities in (56) and (58) follows from Lemma 8.

Since $\bar{G}_v + \bar{H}_1 \bar{M}_h \bar{G}_y$ is causal and in $L^\infty$, it is in $H^\infty$. By Lemma 5, this further implies $\bar{G}_v + \bar{H}_1 \bar{M}_h \bar{G}_y \in H^2$ as its STPBC has no feed through term. Using Lemma 6, we have

$$
\|\bar{G}_v + \bar{H}_1 \bar{M}_h \bar{G}_y\|_{H^2}^2 = \frac{1}{\bar{h}} \|\bar{D}\|_{H^2}^2 + \|\bar{Y}\|_{H^2}^2
$$

where

$$
\bar{Y} = \begin{pmatrix} T_s e^{Ah} Y_r B_c \\ C_c \\ 0 \end{pmatrix} = \begin{pmatrix} A_{ms} & Q_m & \bar{B}_{ms} \\ 0 & e^{Ah} & \bar{B}_{mu} \\ C_{ms} & C_{vu} \end{pmatrix}
$$

where $Q_m := \begin{pmatrix} M_{1w} e^{Ah} \\ 0 \\ 0 \end{pmatrix}$. Now, $\bar{Y} \in H^2$ as $\bar{G}_v + \bar{H}_1 \bar{M}_h \bar{G}_y \in H^2$ and $\|\bar{D}\|_{H^2}^2$ is finite.

Since $A_{ms}$ is Schur and $e^{-Ah}$ has all its poles in the closed unit disk of the complex plane, therefore the Sylvester equation (61) has a unique solution $X_m$. Now, applying a state transform $T := \begin{pmatrix} I - X_m \\ 0 \end{pmatrix}$, we have

$$
\bar{Y} = \begin{pmatrix} A_{ms} & 0 & \bar{B}_{ms} - X_m \bar{B}_{mu} \\ 0 & e^{Ah} & \bar{B}_{mu} \\ C_{ms} & C_{ms} X_m + C_{mu} \end{pmatrix}
$$

Since $\bar{Y} \in H^2$, $\begin{pmatrix} e^{Ah} \\ C_{ms} X_m + C_{mu} \end{pmatrix} \bar{B}_{mu} = 0$. Therefore,

$$
\bar{Y} = \begin{pmatrix} A_{ms} \bar{B}_{ms} - X_m \bar{B}_{mu} \\ 0 \end{pmatrix}
$$

This shows that the STPBC of the system $\bar{G}_v + \bar{H}_1 \bar{M}_h \bar{G}_y$ given in (55) contains unobservable or uncontrollable poles that lie in the region $|z| \geq 1$ of the complex plane.

Note that $A_{ms}$ is Schur. The rest of the proof is standard (see e.g. [23, lemma 5.3.31]). □
Proof (Lemma 21):
For a given integer $k$, we have $\|z^k\tilde{G}\|_{L^2} = \|\tilde{G}\|_{L^2}$ for a system $\tilde{G}$. Therefore,
\[
\|\text{proj}_{L^2 \setminus z^kH^2}(\tilde{H}^* \tilde{N}_v \tilde{N}_y - \tilde{V})\|_{L^2} = \|\tilde{P}\|_{L^2}
\]
where
\[
\tilde{P}(z) := \frac{1}{z^{(k+1)}} \text{proj}_{L^2 \setminus z^kH^2}(\tilde{H}^* \tilde{N}_v \tilde{N}_y - \tilde{V}).
\]
Let $\tilde{Y} := \tilde{H}^* \tilde{N}_v \tilde{N}_y - \tilde{V}$.

Then using (51) and (52), the adjoints are
\[
\tilde{Y}^* (z) = \begin{bmatrix} -A^*_p & -C^*_p \\ B^*_p & 0 \end{bmatrix} \begin{bmatrix} zI - \Omega^*_p \end{bmatrix}^+ \mathcal{J}_0^{*},
\]
\[
\tilde{P}^* (z) = \begin{bmatrix} -A^*_p & -(C_p(e^{-A_p \tau} \Omega_p)^{1+1})^* \\ B^*_p & 0 \end{bmatrix} \begin{bmatrix} zI - \Omega^*_p \end{bmatrix}^+ \mathcal{J}_0^{*}.
\]

The samplers $\tilde{Y}$ and $\tilde{P}$ are anti-causal systems in $L^\infty$, therefore their conjugates are holds in $H^\infty$. Using Lemma 5 this further implies that holds $\tilde{Y}^*$ and $\tilde{P}^*$ are in $H^2$ as well. Therefore, $\|\tilde{Y}^*\|_{H^2} = \|\tilde{Y}\|_{L^2}$ and $\|\tilde{P}^*\|_{H^2} = \|\tilde{P}\|_{L^2}$ can be obtained by Lemma 7.

To calculate the norms we need to evaluate the integral
\[
\tilde{B}_p \tilde{B}^*_p = \int_0^h e^{-A_p \tau} \tilde{B}_p(\tau) B_p(\tau)^* e^{-A_p \tau} d\tau.
\]

The above integral is not straightforward as $B(\tau)$ is not constant but a function of $\tau$. The rest of the proof is mainly devoted to evaluation of the above integral. Using [10, Theorem 1] and $A_p = \begin{bmatrix} A^*_H & C^*_H C_v X \\ 0 & A^*_L \end{bmatrix}$, we have that
\[
e^{-A_p \tau} = e^{\begin{bmatrix} A^*_H & C^*_H C_v X \\ 0 & A^*_L \end{bmatrix} \tau} = \begin{bmatrix} e^{A^*_H \tau} T_1(\tau) \\ 0 \\ e^{A^*_L \tau} \end{bmatrix},
\]
where $T_1(\tau) := \int_0^\tau e^{A^*_H (\tau - \sigma)} C^*_H C_v X e^{A^*_L \sigma} d\sigma$. Recall $A_L := A + L C_y$. Therefore, using $P(\tau)$ given in Lemma 17 as
\[
P(\tau) = e^{-A^*_H \tau} P_0 e^{-A^*_\tau} - \int_0^\tau e^{-A^*_H \sigma} C^*_H C_v e^{-A^*_\sigma} d\sigma,
\]
and $B_p(\tau) = \begin{bmatrix} -P(\tau) L Z_{y}^{-1} \\ (Z_y C_y)^* \end{bmatrix}$, we have that
\[
e^{-A_p \tau} B_p(\tau) = e^{\begin{bmatrix} -A^*_H & -C^*_H C_v X \\ 0 & -A^*_L \end{bmatrix} \tau} \begin{bmatrix} -P(\tau) L Z_{y}^{-1} \\ (Z_y C_y)^* \end{bmatrix} = \begin{bmatrix} -P_0 e^{-A^*_\tau} L Z_{y}^{-1} + T_2(\tau) L Z_{y}^{-1} + T_1(\tau)(Z_y C_y)^* \\ e^{A^*_L \tau} (Z_y C_y)^* \end{bmatrix}
\]
where $T_2(\tau) := e^{A^*_H \tau} \int_0^\tau e^{-A^*_H \sigma} C^*_H C_v e^{-A^*_\sigma} d\sigma$. Using [10, Theorem 1] again, we can show that the above is equal to
\[
e^{-A_p \tau} B_p(\tau) = P_z e^{A^*_L \tau} B_z
\]
Using the above, we have that
\[
\int_0^h e^{-A_p \tau} B_p(\tau) B_p(\tau)^* e^{-A_p \tau} d\tau = P_z \left( \int_0^h e^{A^*_L \tau} B_z B^*_z e^{A^*_L \tau} d\tau \right) P^*_z = P_z A^*_{122} (A^*_z, B^*_z) A_{112} (A^*_z, B^*_z) P^*_z
\]
where \( A_{22} \) and \( A_{12} \) are defined in Lemma 8. Now, using Lemma 7 we have
\[
\| Y^\sim \|_{H^2} = \left\| \left( \begin{array}{c} \Omega \rho e^{-A_l^h} - \frac{A_{12}^*}{B^*_p} \\ \frac{C^*_p}{B^*_p} \\
\end{array} \right) \right\|_{H^2},
\]
\[
\| P^\sim \|_{H^2} = \left\| \left( \begin{array}{c} \Omega \rho e^{-A_l^h} - \frac{A_{12}^*}{B^*_p} \\ \frac{C^*_p}{B^*_p} \\
\end{array} \right) \right\|_{H^2}.
\]

The rest of the proof is standard (see e.g. [23, lemma 5.3.31]).

\( \square \)

References