Parameter Identification of Stochastic Diffusion Systems with Unknown Boundary Conditions

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Abstract

This paper treats the filtering and parameter identification for the stochastic diffusion systems with unknown boundary conditions. The physical situation of the unknown boundary conditions can be found in many industrial problems, i.e., the salt concentration model of the river Rhine is a typical example. After formulating the diffusion systems by regarding the noisy observation data near the systems boundary region as the system’s boundary inputs, we derive the Kalman filter and the related likelihood function. The consistency property of the maximum likelihood estimate for the systems parameters is also investigated. Some numerical examples are demonstrated.

I. INTRODUCTION

Estimation and control problems of distributed parameter systems are complex, although physically highly relevant, subjects. Examples include chemical reactors and flexible beams, which are modeled by parabolic and hyperbolic partial differential equations respectively. In most situations the boundary conditions are clearly specified from physical considerations. However, in some specific situations, boundaries have to be set arbitrarily and are only known through measurements. This makes the boundary conditions inherently noisy. One such problem was studied by Bagchi et. al. [1], that arose is modeling the salt concentration of the river de Waal that represents the part of Rhine flowing through the Netherlands. To pre-determine the effect of any calamity in the Rhine before it enters the Netherlands on the quality of water to be stored in reservoirs downstream in Gorinchem, salt concentration of de waal has been modeled from Lobith (where Rhine enters the Netherlands) to Nieuw Merweerd at the estuary of the North sea. Two monitoring stations, one at Lobith and the other at Gorinchem were used for the modeling purpose. The measurements at Lobith provided the noisy boundary conditions mentioned earlier.

The solution given in [1] was based on discretization, following which the model parameters were estimated by maximizing a quasi-likelihood function. The basic problem of establishing existence of solution of the modeled partial differential equation subject to the noisy boundary condition remained unresolved. Another attempt to solve the problem was made by Aihara and Bagchi [2], in which the authors worked with the continuous model but sidestepped the existence issue by taking the boundary conditions as deterministic, but unknown functions in appropriate spaces. The problem is then transformed into optimal control problems for partial differential equations and leads to horrendous sets of equations which are very difficult to solve.

The reason behind these unsatisfactory formulations is the difficulty of studying the original problem head on. There are two reasons behind this. One is establishing existence of solution of (stochastic) partial differential equations with noisy boundary conditions. The other is to appropriately define a likelihood functional whose maximization would lead to appropriate estimates of the model parameters. To the best knowledge of the authors, these two problems are resolved in this paper for the first time.

The paper is organized as follows: in Sec. 2, we mathematically reformulate the salt concentration problem as the stochastic parabolic systems with noisy boundary inputs. The Kalman filter and the likelihood function are derive in Sec.3. The parameter estimation problem is proposed by using the maximum likelihood estimation (MLE) in Sec.4. The time asymptotic behaviors of the consistency property of MLE is also studied as the number of monitoring station on boundary points. The Section 5 is devoted to show some simulation results to show the feasibility of the proposed algorithm.

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II. PROBLEM STATEMENTS

We consider the following stochastic heat diffusion equation:

\[ du(t, x) = a \frac{\partial^2 u(t, x)}{\partial x^2} dt + V \frac{\partial u(t, x)}{\partial x} dt + dw(t, x), \quad \text{for } x \in \text{some region} \]

where \( a > 0 \) and \( w(t, x) \) denotes the two-dimensional Brownian motion process (BMP) with

\[ E \{ w(t, x_1)w(t, x_2) \} = q(x_1, x_2)t. \]

Although there exist many situations that the spatial variable \( x \) is defined in a bounded region with boundary conditions, in our salt concentration problem it is difficult to set the spatial region and boundary conditions. We only observe the value \( u \) at some fixed points. For simplicity we set

\[
\begin{align*}
(2)
\quad & dy_1^b(t) = u(t, 0)dt + \sigma_0 dv_0^b(t) \\
(3)
\quad & dy_2^b(t) = u(t, 1)dt + \sigma_1 dv_1^b(t) \\
(4)
\quad & dy_m(t) = \int_{G_o} h(x)u(t, x)dxdt + \sigma_m dv_m(t),
\end{align*}
\]

where \( v_0^b, v_1^b \) and \( v_m \) are mutually independent BMPs , \( G_o \subset \mathbb{R}^d \) and \( h(x) \) is a some smooth function.

Now by using (2) and (3), we construct the boundary conditions on \( x = 0, 1 \), i.e.,

\[
\begin{align*}
(5)
\quad & u(t, x) = u_0(x) + \int_0^t a \frac{\partial^2 u(s, x)}{\partial x^2} ds + \int_0^t V \frac{\partial u(s, x)}{\partial x} ds + w(t, x) \\
(6)
\quad & \int_0^t u(s, 0)ds = y_0(t) - \sigma_0 v_0(t) \\
(7)
\quad & \int_0^t u(s, 1)ds = y_1(t) - \sigma_1 v_1(t)
\end{align*}
\]

with the observation mechanism

\[ dy_m(t) = \int_{G_o} h(x)u(t, x)dxdt + \sigma_m dv_m(t). \]

In this formulation, \( y_0(t) \) and \( y_1(t) \) are used as the fixed boundary inputs and under \( \mathcal{F}_t^{y_m} \) we construct a likelihood functional to identify \( a \) and \( V \). To do this, first we need to formulate the above reconstructed systems where the boundary conditions do not include the integral term with respect to the time variable \( t \).

Setting

\[
\begin{align*}
(8)
\quad & \tilde{u}(t, x) = A^{-1}u(t, x) - x(y_1^b(t) - \sigma_1 v_1^b(t)) - (1 - x)(y_0^b(t) - \sigma_0 v_0^b(t)),
\end{align*}
\]

and noting that \(-a\frac{\partial^2}{\partial x^2}(\tilde{u}(t, x) + x(y_1^b(t) - \sigma_1 v_1^b(t)) + (1 - x)(y_0^b(t) - \sigma_0 v_0^b(t))) = u(t, x)\), we have

\[
\begin{align*}
(9)
\quad & \tilde{u}(t, x) - \int_0^t \left( a \frac{\partial^2}{\partial x^2} + V \frac{\partial}{\partial x} \right) \left( \tilde{u}(s, x) + x(y_1^b(t) - \sigma_1 v_1^b(t)) + (1 - x)(y_0^b(t) - \sigma_0 v_0^b(t)) \right) ds = \tilde{u}_0(x) + \tilde{w}(t, x)
\end{align*}
\]

with the boundary conditions (from (8))

\[
\begin{align*}
(10)
\quad & \begin{cases}
\tilde{u}(t, 1) + y_1^b(t) - \sigma_1 v_1^b(t) = 0 \\
\tilde{u}(t, 0) + y_0^b(t) - \sigma_0 v_0^b(t) = 0,
\end{cases}
\end{align*}
\]

where

\[ A^{-1}\phi(x) = \sum_{i=1}^{\infty} \frac{1}{a(i\pi)^2} \sqrt{2} \sin(i\pi x) \int_0^1 \sqrt{2} \sin(i\pi x) \phi(x) dx. \]
We also have
\[ y_m(t) = \int_0^t \int_{G_o} \tilde{h}(x) \tilde{u}(s, x) dx ds + \sigma_m v_m(t), \]
where
\[ \tilde{h}(x) = -a \frac{\partial^2 h(x)}{\partial x^2}, \]
and here we assume that \( h \) is twice continuously differentiable with \( \tilde{h}(x) = 0, \frac{\partial h(x)}{\partial x} = 0 \) on the boundary \( \partial G_o \).
The above derivation is demonstrated in Appendix-A.

### III. FILTERING PROBLEM

As illustrated in Fig.1, it is convenient to transform the system with the robust boundary conditions to the stochastic ordinary differential equation form in some function spaces.

Before deriving the Kalman filter, we need to show the existence of a unique solution of the transformed system (9) with (10). We work in the following Sobolev spaces \(^1\):
\[ V = H^1(0, 1) \subset H = L^2(0, 1) \subset V' = \text{dual of } V. \]

It is not convenient that the system (9) and boundary conditions (10) are separately given. For introducing the following weak integral form, the boundary inputs are included in the interior region of the system and the filter and covariance equations are easily derived from the Gaussian property \((\tilde{u}(t), \tilde{\phi})\).

Now choosing \( \tilde{\phi} \in H_0^1 \cap H^2 \), multiplying this to (9) and integrating by parts, (9) with (10) is converted to the following form:
\[ (\tilde{u}(t), \tilde{\phi}) + \int_0^t (\tilde{u}(s) + x(y_{1b}(s) - \sigma_1 v_{1b}(s))) \]
\[ + (1-x)(y_{0b}(s) - \sigma_0 v_{0b}(s)), (A + B^*) \tilde{\phi} ) ds = (\tilde{u}_o, \tilde{\phi}) + (\tilde{w}(t), \tilde{\phi}), \]
\[ \text{where } A = \int_0^1 \int_{G_o} h(x) u(t, x) dx dx + \sigma_m r_m(t), \]

\[ V = H^1(0, 1) \subset H = L^2(0, 1) \subset V' = \text{dual of } V. \]

1We denote \( H^m(0, 1) \) as the \( m \)-th order Sobolev space and \( H_0^m \) means that \( \phi \in H^2(0, 1) \) with \( \phi(0) = \phi(1) = 0 \) and \( \frac{\partial \phi}{\partial x} = 0 \) on \( x = 0, 1. \) \( (\phi_1, \phi_2) \) denotes the inner product in \( H \) with the norm \( | \cdot |. \)
for $\tilde{\phi} \in H_0^1 \cap H^2$ where
\[ A = -a \frac{\partial^2(\cdot)}{\partial x^2} \quad \text{and} \quad B = -V \frac{\partial(\cdot)}{\partial x} . \]

**Theorem 1:** We assume that

(A-1): $y_1^b, y_0^b \in C([0, T]; R^1)$, a.s.

(A-2): $Tr\{\tilde{Q}\} < \infty$,

where $\tilde{Q} = A^{-1}(A^{-1}Q)^*$ for $Q = \int_0^1 q(x, y)(\cdot)dy$ and

(A-3): $\tilde{u}_o \in L^2(\Omega; H)$.

The system (11) has a unique solution:
\[ \tilde{u} \in L^2(\Omega; L^\infty([0, T]; H)). \]

Furthermore assuming that

(A-4): $h \in H_0^2(G_o)$

the signal part of $y_m$ is well defined, i.e.,
\[ E\{\int_0^t | \int_{G_o} a \frac{\partial^2 h(x)}{\partial x^2} \tilde{u} dx |^2 dt\} < \infty. \]

**Theorem 2:** Instead of (A-1), we set the strong assumption:

(A-1)': $y_1^b$ and $y_0^b$ are given by (2) and (3) with
\[ E\{\int_T (|u(0, t)|^2 + |u(1, t)|^2) dt\} < \infty. \]

Hence (11) has a unique solution:
\[ \tilde{u} \in L^2(\Omega; C([0, T]; H)) \cap L^2([0, T]; V)). \]

The proofs of these theorems are shown in Appendix-B.

In our formulation, $y_1^b$ and $y_0^b$ are set as the known boundary inputs and we need to estimate $\tilde{u}, v_1^b$ and $v_0^b$ under $F_{\tilde{u}, v}$. Set the extended state,
\[ \tilde{w}(t, x) = [\tilde{u}(t, x) \ v_1^b(t) \ v_0^b(t)]', \]

where $v_1^b$ and $v_0^b$ are BMPs.

The extended state becomes
\[
\begin{align*}
d & \begin{pmatrix} (\tilde{u}(t), \phi) \\
v_1^b(t) \\
v_0^b(t) \end{pmatrix} + \begin{pmatrix} 1 & -\sigma_1(x, (A + B^*)\phi) & -\sigma_0(1 - x, (A + B^*)\phi) \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (\tilde{u}(t), (A + B^*)\phi) \\
v_1^b(t) \\
v_0^b(t) \end{pmatrix} dt \\
& + \begin{pmatrix} (xy_1^b(t) + (1 - x)y_0^b(t), (A + B^*)\phi) \\
0 \\
0 \end{pmatrix} dt = d \begin{pmatrix} (\tilde{w}(t), \phi) \\
v_1^b(t) \\
v_0^b(t) \end{pmatrix} 
\end{align*}
\]

(12)

with

(13) $dy^m(t) = (1 \ 0 \ 0) \begin{pmatrix} \int_{G_o} \tilde{h}(x)\tilde{u}(t, x)dx \\
v_1^b(t) \\
v_0^b(t) \end{pmatrix} dt + \sigma_m dv^m(t)$. 

\[ dy_m(t) = (1 \ 0 \ 0) \begin{pmatrix} \int_{G_o} \tilde{h}(x)\tilde{u}(t, x)dx \\
v_1^b(t) \\
v_0^b(t) \end{pmatrix} dt + \sigma_m dv^m(t) \]
This is a linear Gaussian problem and the Kalman filter can be easily derived. Denoting \( \hat{z} = E\{|\mathcal{F}_{t}^{m}\} \), we have
\[
d\begin{pmatrix}
(\hat{u}(t), \phi) \\
\hat{v}_{0}^{t}(t) \\
\hat{v}_{0}^{t}(t)
\end{pmatrix} + \begin{pmatrix}
1 - \sigma_{1}(x, (A + B^{*})\phi) & -\sigma_{0}(1 - x, (A + B^{*})\phi) \\
0 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
(\hat{u}(t), (A + B^{*})\phi) \\
\hat{v}_{0}^{t}(t) \\
\hat{v}_{0}^{t}(t)
\end{pmatrix} dt
+ \begin{pmatrix}
(xy_{0}^{t}(t) + (1 - x)y_{0}^{t}(t), (A + B^{*})\phi) \\
0 \\
0
\end{pmatrix} dt
\]
(14)
\[
= \begin{pmatrix}
(\int_{G_{o}} \hat{p}_{11}(t, x, y) \hat{h}(y) dy, \phi) \\
\int_{G_{o}} \hat{p}_{21}(t, x) \hat{h}(x) dx \\
\int_{G_{o}} \hat{p}_{31}(t, x) \hat{h}(x) dx
\end{pmatrix} \frac{1}{\sigma_{m}^{2}} (dy_{m}(t) - \int_{G_{o}} \hat{h}(x) \hat{u}(t, x) dx dt),
\]
where the covariance operators \( \hat{p}_{11}(t, x, y) = E\{(\hat{u}(t, x) - \hat{u}(t, x))(\hat{u}(t, y) - \hat{u}(t, y)) | \mathcal{F}_{t}^{m}\}, \hat{p}_{21}(t, x) = E\{(v_{0}^{t}(t) - \hat{v}_{0}^{t}(t))(\hat{u}(t, x) - \hat{u}(t, x)) | \mathcal{F}_{t}^{m}\}, \hat{p}_{31}(t, x) = E\{(v_{0}^{t}(t) - \hat{v}_{0}^{t}(t))(v_{0}^{t}(t) - \hat{v}_{0}^{t}(t)) | \mathcal{F}_{t}^{m}\}, \hat{p}_{23}(t) = E\{(v_{1}^{t}(t) - \hat{v}_{1}^{t}(t))(v_{0}^{t}(t) - \hat{v}_{0}^{t}(t)) | \mathcal{F}_{t}^{m}\} \) and \( \hat{p}_{33}(t) = E\{(v_{0}^{t}(t) - \hat{v}_{0}^{t}(t))(v_{0}^{t}(t) - \hat{v}_{0}^{t}(t)) | \mathcal{F}_{t}^{m}\} \) are given by
\[
\frac{\partial \hat{p}_{21}(t, x)}{\partial t} + \frac{1}{\sigma_{m}} \int_{G_{o}} \hat{h}(z) \hat{p}_{21}(t, z) dz \int_{G_{o}} \hat{h}(y) \hat{p}_{11}(t, y) dy, \phi = 0
\]
(15)
\[
\frac{\partial \hat{p}_{31}(t, x)}{\partial t} + \frac{1}{\sigma_{m}} \int_{G_{o}} \hat{h}(z) \hat{p}_{31}(t, z) dz \int_{G_{o}} \hat{h}(y) \hat{p}_{11}(t, y) dy, \phi = 0
\]
(16)
\[
\frac{dp_{22}(t)}{dt} + \frac{1}{\sigma_{m}^{2}} (\int_{G_{o}} \hat{h}(z) \hat{p}_{21}(t, z) dz)^{2} = 1
\]
(17)
\[
\frac{dp_{33}(t)}{dt} + \frac{1}{\sigma_{m}^{2}} (\int_{G_{o}} \hat{h}(z) \hat{p}_{31}(t, z) dz)^{2} = 1,
\]
(18)
\[
\frac{dp_{23}(t)}{dt} + \frac{1}{\sigma_{m}^{2}} \int_{G_{o}} \hat{h}(z) \hat{p}_{21}(t, z) dz \int_{G_{o}} \hat{h}(z) \hat{p}_{21}(t, z) dz = 0
\]
(19)
and
\[
\int_{0}^{1} \int_{0}^{1} \frac{\partial \hat{p}_{11}(t, x, y)}{\partial t} \phi(y) dy \psi(x) dx d\theta
\]
(20)
\[
+ \int_{0}^{1} \int_{0}^{1} \{\hat{p}_{11}(t, x, y) - \sigma_{1} \hat{p}_{21}(t, x) y - \sigma_{0} \hat{p}_{31}(t, x)(1 - y)\}(-a \frac{\partial^{2}}{\partial y^{2}} + V \frac{\partial}{\partial y}) \phi(y) dy \psi(x) dx d\theta
\]
\[
+ \int_{0}^{1} \int_{0}^{1} \{\hat{p}_{11}(t, x, y) - \sigma_{1} x \hat{p}_{21}(t, y) - \sigma_{0}(1 - x) \hat{p}_{31}(t, y)\} \phi(y) dy (-a \frac{\partial^{2}}{\partial x^{2}} + V \frac{\partial}{\partial x}) \psi(x) dx d\theta
\]
\[
+ \int_{0}^{1} \int_{0}^{1} \int_{G_{o}} \hat{h}(z) \hat{p}_{11}(t, x, z) dz \phi(x) \int_{G_{o}} \hat{h}(z) \hat{p}_{11}(t, y, z) dz \psi(y) dy dx dy
\]
\[
= \int_{0}^{1} \int_{0}^{1} \hat{q}(x, y) \phi(x) \psi(y) dx dy
\]
for all $\phi, \psi \in H_0^1 \cap H^2$.

Now the estimate of the original state $u$ is given by

$$\hat{u}(t, x) = A\hat{u}(t, x) = -a\frac{\partial^2}{\partial x^2}\hat{u}(t, x).$$

The partial differential equation form of $\hat{u}(t, x)$ is expressed by

$$d\hat{u}(t, x) - a\frac{\partial^2 \hat{u}(t, x)}{\partial x^2} dt - V\frac{\partial \hat{u}(t, x)}{\partial x} dt - V \{(y_1^b(t) - \sigma_1 \hat{v}_1^b(t)) - (y_0^b(t) - \sigma_0 \hat{v}_0^b(t))\} dt = \int_{G_o} \tilde{h}(y) \hat{p}_{11}(t, x, y) dy \frac{1}{\sigma_m^2} (dy_m(t)) - \int_{G_o} \tilde{h}(x) \hat{u}(t, x) dx dt$$

with the boundary condition:

$$\begin{cases} \hat{u}(1, x) + y_1^b(t) - \sigma_1 \hat{v}_1^b(t) = 0 \\ \hat{u}(0, x) + y_0^b(t) - \sigma_0 \hat{v}_0^b(t) = 0. \end{cases}$$

The estimates $\hat{v}_1^b$ and $\hat{v}_0^b(t)$ are given by their original forms in (14). The gain operators are also given by

$$\frac{\partial \hat{p}_{11}(t, x, y)}{\partial t} = (a\frac{\partial^2}{\partial y^2} + V\frac{\partial}{\partial y})\hat{p}_{11}(t, x, y) - (a\frac{\partial^2}{\partial x^2} + V\frac{\partial}{\partial x})\hat{p}_{11}(t, x, y) + V\{v_1^b(t) + \hat{v}_{11}(t, y) - \sigma_0(\hat{p}_{31}(t, x) + \hat{p}_{31}(t, y))\}$$

$$+ \frac{1}{\sigma_m^2} \int_{G_o} \tilde{h}(y) \hat{p}_{11}(t, x, y) dy \int_{G_o} \tilde{h}(z) \hat{p}_{11}(t, y, z) dz = \hat{q}(x, y),$$

with the boundary conditions

$$\hat{p}_{11}(t, x, 0) = \sigma_0 \hat{p}_{31}(t, x), \quad \hat{p}_{11}(t, x, 1) = \sigma_1 \hat{p}_{21}(t, x),$$

$$\hat{p}_{11}(t, 0, y) = \sigma_0 \hat{p}_{31}(t, y), \quad \hat{p}_{11}(t, 1, y) = \sigma_1 \hat{p}_{21}(t, y),$$

and

$$\frac{\partial \hat{p}_{21}(t, x)}{\partial t} = (a\frac{\partial^2}{\partial x^2} + V\frac{\partial}{\partial x})\hat{p}_{21}(t, x) + \frac{1}{\sigma_m^2} \int_{G_o} \tilde{h}(z) \hat{p}_{21}(t, z) dz \int_{G_o} \tilde{h}(y) \hat{p}_{11}(t, y, x) dy = -\sigma_1 p_{22}(t) + \sigma_0 p_{23}(t)$$

with boundary conditions:

$$\hat{p}_{21}(t, 1) = \sigma_1 p_{22}(t), \quad \hat{p}_{21}(t, 0) = \sigma_0 p_{23}(t)$$

and

$$\frac{\partial \hat{p}_{31}(t, x)}{\partial t} = (a\frac{\partial^2}{\partial x^2} + V\frac{\partial}{\partial x})\hat{p}_{31}(t, x) + \frac{1}{\sigma_m^2} \int_{G_o} \tilde{h}(z) \hat{p}_{31}(t, z) dz \int_{G_o} \tilde{h}(y) \hat{p}_{11}(t, y, x) dy = \sigma_0 p_{33}(t) - \sigma_1 p_{23}(t)$$

with boundary conditions:

$$\hat{p}_{31}(t, 1) = \sigma_1 p_{23}(t), \quad \hat{p}_{31}(t, 0) = \sigma_0 p_{33}(t),$$

and $p_{22}(t)$ and $p_{33}(t)$ are given by (17, 18), respectively.
IV. PARAMETER IDENTIFICATION

For identifying the parameters contained in the system, we need to derive the likelihood function
\( LF(y_m, \theta) \) for \( \theta = [a \ V] \). The likelihood function is given by the Radon-Nikodym derivative of the
measure \( \mathcal{P}_{y_m} \) with respect to the measure \( \mathcal{P}_{v_m} \). This derivative is given by

\[
\frac{d\mathcal{P}_{y_m}}{d\mathcal{P}_{v_m}} = \exp\{\int_0^t \int_{G_o} \tilde{h}(x) \hat{u}(s, x)dx \, dy_m(s) - \frac{1}{2} \int_0^t \int_{G_o} \frac{\hat{h}(x) \hat{u}(s, x)}{\sigma_m} \, dx^2 \, ds\}.
\]

Hence we can identify the parameter \( \theta \) for maximizing the log likelihood function, i.e.,

\[
\hat{\theta} = \arg\max_{\theta} \left\{ \int_0^t \int_{G_o} \tilde{h}(x) \hat{u}(s, x; \theta)dx \, dy_m(s) - \frac{1}{2} \int_0^t \int_{G_o} \frac{\hat{h}(x) \hat{u}(s, x; \theta)}{\sigma_m} \, dx^2 \, ds \right\}.
\]

For the original system form, we also have

\[
\hat{\theta} = \arg\max_{\theta} \left\{ \int_0^t \int_{G_o} \tilde{h}(x) \hat{u}(s, x; \theta)dx \, dy_m(s) - \frac{1}{2} \int_0^t \int_{G_o} \frac{\hat{h}(x) \hat{u}(s, x; \theta)}{\sigma_m} \, dx^2 \, ds \right\}.
\]

A. Consistency Property of MLE

The consistency property of MLE has already been studied in [3], [4], [5]. In these works, the asymptotic
property of MLE as \( t \to \infty \) is mainly checked. To study the consistency property of MLE, we set many
sensors (say M) on each point of the boundaries. The convergence property of the MLE \( \theta_M \) to the true
value \( \theta_0 \) in some sense is mathematically checked as \( M \) and \( t \to \infty \).

The idea of many observations has been initially proposed by [6], [7], [8], [9], [10], when we can
perform many independent experiments. Fortunately, for the distributed systems, it is possible to set many
sensors on the boundaries, whose sensors are naturally perturbed by the independent observation noises.
Hence for the distributed parameter systems we obtain many independent observation data at once without
repeating many independent experiments.

Now we reset the boundary observation mechanisms;\(^2\)

\[
\begin{align*}
dy_0^b(t) &= u(t, 0)dt + \sigma_0 dv_0(t), \\
dy_1^b(t) &= u(t, 1)dt + \sigma_1 dv_1(t)
\end{align*}
\]

where \( \{v_k\}_{i=1}^M \) are mutually independent Brownian motion processes for \( i = 1, 2, 3, \ldots, M, k = 1, 2 \). Averaging these data, we use the following boundary observation data:

\[
\begin{align*}
y_0^{b,M}(t) &= \frac{1}{M} \sum_{i=1}^M y_0^b(t), \\
y_1^{b,M}(t) &= \frac{1}{M} \sum_{i=1}^M y_1^b(t)
\end{align*}
\]

with

\[
\begin{align*}
v_0^{b,M}(t) &= \frac{1}{M} \sum_{i=1}^M v_0^b(t), \text{ and } v_1^{b,M}(t) = \frac{1}{M} \sum_{i=1}^M v_1^b(t).
\end{align*}
\]

For the observation in the inner region, we assume that \( \tilde{h}(x) \) is independent of \( a \) for simplicity and
extend this function as zero outside \( G_o \) smoothly, i.e., \( y_m(t) \) is denoted by

\[
dy_m(t) = H\hat{u}(t)dt + \sigma dv_m(t),
\]

\(^2\)In this paper, we do not consider the width of river and hence on each boundary the same signal is observed with independent noises.
where

\[ H\phi = \int_0^1 \hat{h}(x)\phi(x)dx. \]

The consistency property is usually studied under the assumption that the system state has reached the stationary state in [3, 4, 5], i.e., covariance operators are replaced by algebraic forms. In this paper, to realize the stationary state we replace the operator \( A(\theta) \) as \( A_\delta(\theta) \) for a small positive \( \delta \), i.e.,

\[ A_\delta(\theta) = -a \frac{\partial^2(\cdot)}{\partial x^2} + \delta(\cdot), \]

in the covariance operators (30), (33) and (35).

Assume that

(A-5): \( \sup_{t \in [0, \infty)} (E\{|u(0, t)|^2\} + E\{|u(1, t)|^2\} \leq C. \)

The Kalman filter under this situation becomes

\[
\begin{align*}
\frac{d\hat{u}(t; \theta)}{dt} + (A_\delta(\theta) + B^*(\theta))\phi_1 + (A_\delta(\theta) + B^*(\theta))\phi_2 - \sigma_1 x\hat{v}_1^{bM}(t; \theta) - \sigma_0(1 - x)\hat{v}_0^{bM}(t; \theta), (A(\theta) + B^*(\theta))\phi dt \\
+ (x y_1^{bM}(t) + (1 - x) y_0^{bM}(t), (A(\theta) + B^*(\theta))\phi dt \\
= (\phi, \tilde{P}_{11}(t; \theta))\frac{1}{\sigma^2}(dy_m(t) - H\hat{u}(t; \theta)dt)
\end{align*}
\]

where we denote \( \tilde{P}_{11}(t; \theta) = \int_G \tilde{p}_{11}(x, y)(\cdot)dy \) and for \( \phi_1, \phi_2 \in H_0^1 \cap H^2 \)

\[ \frac{d\hat{P}_{11}(t; \theta)}{dt} \]

\[ \frac{d\tilde{p}_{k1}(t; \theta)}{dt} \]

\[ \frac{d\tilde{p}_{kk}(t; \theta)}{dt} + \frac{1}{\sigma^2}(\hat{p}_{k1}(t; \theta), H^*H\hat{P}_{k1}(t; \theta)) = \frac{1}{M}, \]

\[ \frac{d\tilde{p}_{21}(t; \theta)}{dt} + \frac{1}{\sigma^2}(\hat{p}_{31}(t; \theta), H^*H\hat{p}_{21}(t; \theta)) = 0. \]

Now in order to check the consistency property of MLE, we define the exact innovation process for \( \theta^0 = [a^0 \; V^0] \) true value;

\[ z(t; \theta^0) = y_m(t) - \int_0^t H\hat{u}(s; \theta^0)ds. \]

The Kalman filter is represented by

\[ \frac{d\hat{u}(t; \theta)}{dt} + (\hat{u}(t; \theta) - \sigma_1 x \tilde{v}_1^{bM}(t; \theta) - \sigma_0(1 - x)\tilde{v}_0^{bM}(t; \theta), (A(\theta) + B^*(\theta))\phi dt \\
+ (x y_1^{bM}(t) + (1 - x) y_0^{bM}(t), (A(\theta) + B^*(\theta))\phi dt \\
= (\phi, \tilde{P}_{11}(t; \theta))\frac{1}{\sigma^2}(dz(t; \theta^0) + H\hat{u}(t; \theta^0) - \hat{u}(t; \theta))dt) \]

\[ \text{This representation is only used for proving the consistency property of MLE.} \]
and

\[ \hat{v}^k_{KM}(t; \theta) = \int_0^t (\hat{p}_{k+11}(s; \theta), H^*) \frac{1}{\sigma^2} (dz(s; \theta^o) + H(\hat{u}(s; \theta^o) - \hat{u}(s; \theta))ds) \quad k = 1, 2. \]

Now likelihood functional is also represented by

\[ \frac{d\mathcal{P}_{y_m, \theta}}{d\mathcal{P}_{y_m, \theta^o}} = \exp \left\{ -\frac{1}{2\sigma^2} \left( \int_0^t (H(\hat{u}(s; \theta) - \hat{u}(s; \theta^o)))^2 ds \right. \right. \]
\[ \left. \left. - 2 \int_0^t (H(\hat{u}(s; \theta) - \hat{u}(s; \theta^o)))dz(s; \theta^o) \right\} \right. \]

To apply the useful lemma given by Borkar and Bagchi, we assume that

(A-6): Unknown parameters \(a\) and \(V\) satisfy

\[ a_m \leq a \leq a_M, \quad \text{and} \quad V_m \leq V \leq V_M, \]

where these lower and upper bounds are \textit{a priori} known and

\[ \tilde{h} \in H^1_0(0, 1) \cap H^2(0, 1). \]

We need the following propositions:

\textbf{Proposition 1:} For setting

\[ \frac{t_f^3}{M} = C_f = \text{Constant}, \]

there exists an universal constant \(C\) which is independent of \(t_f, M\) and \(\theta\);

\[ \sup_{t \in T} |p_{23}(t; \theta)| \leq \frac{C_f}{t_f}, \quad \sup_{t \in T} |p_{33}(t; \theta)| \leq \frac{C_f}{t_f} \]
\[ \int_0^{t_f} |H \tilde{p}_{21}(s, x; \theta)|^2 ds \leq \sigma \frac{C_f}{t_f}, \quad \int_0^{t_f} |H \tilde{p}_{31}(s, x; \theta)|^2 ds \leq \sigma \frac{C_f}{t_f} \]
\[ \sup_{t \in T} |p_{23}(t; \theta)| \leq \frac{C_f}{t_f}, \quad \sup_{t \in T} |\tilde{p}_{21}(t; \theta)| \leq \frac{C_f}{t_f}, \quad \sup_{t \in T} |\tilde{p}_{31}(t; \theta)| \leq \frac{C_f}{t_f} \]
\[ \sup_{t \in T} |\tilde{P}_{11}(t; \theta)|^2_{HS} \leq C \]

where \(|P|_{HS}^2 = [P]^2 = [P, P]\) for a Hilbert-Schmidt operator \(P\) and \([., .]\) denotes its inner product.

\textbf{Proposition 2:} Denoting \(\nabla_{\theta} f(\theta) = \left[ \frac{\partial f(\theta)}{\partial \theta_1}, \frac{\partial f(\theta)}{\partial \theta_2} \right]^T\), for

\[ \frac{t_f^3}{M} = C_f (\text{Constant}), \]

from (A-6) we have for \(\ell = 1, 2\)

\[ \sup_{t \in T} \left\{ |\nabla_{\theta_{1}} \tilde{P}_{11}(t; \theta) \tilde{h}|^2 + |\nabla_{\theta_{1}} \tilde{p}_{21}(t, x; \theta)|^2 + |\nabla_{\theta_{1}} \tilde{p}_{31}(t, x; \theta)|^2 \right\} \leq C \]

where \(C\) is independent of \(t_f, M\) and \(\theta\).

The exact derivations of these propositions are listed in Appendix-C.

Now we state the main consistency property:

\textbf{Theorem 3:} We assume (A-1)',(A-2) \sim (A-6). Let \(\hat{\theta}\) be the MLE. Hence

\[ \lim_{M \to \infty} \lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} (H(\hat{u}(s; \theta) - \hat{u}(s; \theta^o)))^2 ds = 0. \quad \text{a.s.} \]
\[ \frac{t_f^3}{M} = C_f (\text{Constant}) \]

\textbf{Proof:} From Propositions 1 and 2, we get

\[ E\{H(\hat{u}(t; \theta) - \hat{u}(t; \theta^o))^2\} \leq C|\theta - \theta^o|^2. \]
See Appendix-D for detail derivations of (46). Hence from Lemma 4.12 in Lipster and Shiryaev [11],

\[
E \left\{ \left( \frac{1}{t} \int_0^t H(\hat{\theta}(s; \theta) - \hat{\theta}(s; \theta^o))dz(s; \theta^o) \right)^{2m} \right\}
\]

\begin{align*}
&\leq (m(2m-1))^{m^2m-1} \frac{1}{t^{2m}} \int_0^t E \left\{ \left( (H(\hat{\theta}(s; \theta) - \hat{\theta}(s; \theta^o))^2 \right)^m \right\} ds \\
&\leq (m(2m-1))^{m^2m-1} \frac{1}{t^{2m}} \int_0^t E \left\{ \left| \hat{\theta}(s; \theta) - \hat{\theta}(s; \theta^o) \right|^{2m} \right\} ds
\end{align*}

Noting that \( \hat{\theta} \) is Gaussian, we have

\[
E \{(\hat{\theta}(s; \theta) - \hat{\theta}(s; \theta^o))^{2m}\} = 1 \cdot 2 \cdots (2m - 1)(E\{(\hat{\theta}(s; \theta) - \hat{\theta}(s; \theta^o))^2\})^m.
\]

Hence

\[
E \left\{ \left( \frac{1}{t} \int_0^t H(\hat{\theta}(s; \theta) - \hat{\theta}(s; \theta^o))dz(s; \theta^o) \right)^{2m} \right\} \leq C|\theta - \theta^o|^{2m} t^{2m}. 
\]

From the crucial lemma by Borkar and Bagchi [3], we get

\[
\lim_{M \to \infty} \lim_{t_f \to \infty} \frac{1}{t_f} \left| \int_0^{t_f} H(\hat{\theta}(s; \hat{\theta}) - \hat{\theta}(s; \theta^o))dz(s; \theta^o) \right| = 0 \quad \text{a.s.}
\]

\[
\frac{v_3}{s} = \text{Constant}
\]

Noting that the MLE \( \hat{\theta} \) satisfies

\[
\frac{1}{t_f} \int_0^{t_f} H(\hat{\theta}(s; \hat{\theta}) - \hat{\theta}(s; \theta^o))dz(s; \theta^o) \geq \frac{1}{t_f} \int_0^{t_f} (H(\hat{\theta}(s; \hat{\theta}) - \hat{\theta}(s; \theta^o))^2 ds \geq 0,
\]

(45) can be derived.

\section{V. Simulation studies}

Before performing our simulation studies, we should mention that the robust forms derived in the previous section are not easy to be numerically realized by using the well-known finite difference scheme, because we need to differentiate \( \hat{\theta}(t, x) \) with respect to \( x \). Hence we transform the robust forms into the original state \( \hat{\theta}(t, x) \). Here we present these forms. The derivations are not difficult but very tedious. So we will list up these derivations in Appendix-E.

The original form of the estimator (14) becomes

\[
d\hat{\theta}(t, x) - a \frac{\partial^2 \hat{\theta}(t, x)}{\partial x^2} dt - V \frac{\partial \hat{\theta}(t, x)}{\partial x} dt = \\
\int_{G_o} h(z)p_1(t, x, z)dz \frac{1}{\sigma_m} (dy_m(t) - \int_{G_o} h(x)\hat{\theta}(t, x)dx dt)
\]

with the boundary condition:

\[
\begin{align*}
\left\{ \int_0^t \hat{\theta}(s, 1) ds &= y_1^b(t) - \sigma_1 \hat{v}_1^b(t) \\
\int_0^t \hat{\theta}(s, 0) ds &= y_0^b(t) - \sigma_0 \hat{v}_0^b(t)
\end{align*}
\]

where

\[
\begin{align*}
d\hat{v}_1^b(t) &= \int_{G_o} h(x)p_2(t, x)dx \frac{1}{\sigma_m} (dy_m(t) - \int_{G_o} h(x)\hat{\theta}(t, x)dx dt) \\
d\hat{v}_0^b(t) &= \int_{G_o} h(x)p_3(t, x)dx \frac{1}{\sigma_m} (dy_m(t) - \int_{G_o} h(x)\hat{\theta}(t, x)dx dt)
\end{align*}
\]
and gains $p_{11}, p_{21}$ and $p_{31}$ are given by

$$
\frac{\partial p_{11}(t, x, y)}{\partial t} - (a \frac{\partial^2}{\partial y^2} + V \frac{\partial}{\partial y}) p_{11}(t, x, y) - (a \frac{\partial^2}{\partial x^2} + V \frac{\partial}{\partial x}) p_{11}(t, x, y) \\
+ \frac{1}{\sigma^2_m} \int_{G_o} h(z) p_{11}(t, x, z) dz \int_{G_o} h(z) p_{11}(t, y, z) dz = q(x, y),
$$

with the boundary conditions

$$
p_{11}(t, x, 1) = -\frac{V}{a} \int_0^1 z p_{11}(t, x, z) dz + \frac{1}{2} a \sigma^2 \frac{\partial \delta(x - 1)}{\partial x},
$$

$$
p_{11}(t, x, 0) = \frac{V}{a} \int_0^1 (1 - z) p_{11}(t, x, z) dz + \frac{1}{2} a \sigma^2 \frac{\partial \delta(x)}{\partial x},
$$

$$
p_{11}(t, 1, y) = -\frac{V}{a} \int_0^1 z p_{11}(t, x, z) dz + \frac{1}{2} a \sigma^2 \frac{\partial \delta(x - 1)}{\partial x},
$$

$$
p_{11}(t, 0, y) = \frac{V}{a} \int_0^1 (1 - z) p_{11}(t, x, z) dz + \frac{1}{2} a \sigma^2 \frac{\partial \delta(x - 1)}{\partial x},
$$

and

$$
\frac{\partial p_{21}(t, x)}{\partial t} - (a \frac{\partial^2}{\partial x^2} + V \frac{\partial}{\partial x}) p_{21}(t, x) \\
+ \frac{1}{\sigma^2_m} \int_{G_o} h(z) p_{21}(t, x, z) dz \int_{G_o} h(y) p_{11}(t, y, x) dy = 0
$$

with boundary conditions:

$$
p_{21}(t, 1) = -\sigma_1 - \frac{V}{a} \int_0^1 x p_{21}(t, x) dx,
$$

$$
p_{21}(t, 0) = \frac{V}{a} \int_0^1 (1 - x) p_{21}(t, x) dx
$$

and

$$
\frac{\partial p_{31}(t, x)}{\partial t} - (a \frac{\partial^2}{\partial x^2} + V \frac{\partial}{\partial x}) p_{31}(t, x) \\
+ \frac{1}{\sigma^2_m} \int_{G_o} h(x) p_{31}(t, x) dx \int_{G_o} h(z) p_{11}(t, z, x) dz = 0
$$

with boundary conditions:

$$
p_{31}(t, 1) = -\frac{V}{a} \int_0^1 x p_{31}(t, x) dx,
$$

$$
p_{31}(t, 0) = -\sigma_0 - \frac{V}{a} \int_0^1 (1 - x) p_{31}(t, x) dx.
$$

A. Filtering results

Now we shall present our simulation results. First we set the big spatial region as $-1 < x < 2$. Our world is set as $0 < x < 1$. Initially a pollution exists outside our region ,i.e.,

The true system state $u(t, x)$ is simulated by using the finite difference scheme for $a = 0.01, V = 0.1, \delta x = 0.02, \delta t = 0.001$. The noise kernel $q$ is approximated by

$$
q(x, y) \sim \sigma^2 \sum_{k=1}^{20} \sin(k \frac{x \pi}{\max(x) - \min(x)}) \sin(k \frac{y \pi}{\max(y) - \min(y)})
$$
Fig. 2. Initial state $u(0, x)$ for $\sigma = 0.01$. The simulated state $u$ is shown in Fig. 2.

We observe this state at our boundaries $x = 1, x = 0$ with observation noises as shown in Figs. 3 and 4 for $\sigma_0 = 0.004, \sigma_1 = 0.004$. 
Now at the three points $x = 0.1, 0.32, 0.98$, we observe the state with $\sigma_m = 0.004$ where we approximate

$$\int_{G_o} h(x) u(t, x) dx \sim [u(t, 0.1) \ u(t, 0.32) \ u(t, 0.98)]'.$$

Fig. 4. Boundary observations
Before showing our estimated results, we will present the true system state for $0 < x < 1$ in Fig. 3.5.
Finally our estimated state is demonstrated in Fig. 3.6.

We also show the true and estimated states at the points $x = 0.1, 0.32, 0.9$. 
Fig. 8. True and estimated states at $x = 0.1$

Fig. 9. True and estimated states at $x = 0.32$
B. Identification

Finally we perform the MLE for the systems parameters $a$ and $V$ to maximize the log likelihood. To find the MLE, we used the Generic algorithm which is found in the MATLAB optimization toolbox. We set the parameters bounds as

$$0.001 < a < 0.05 \quad 0.01 < V < 0.5$$

The initial guesses are set as

$$\hat{a} = 0.005 \quad \hat{V} = 0.25.$$  

We also set the generations as 10 and populations size as 20. The final estimated values are

$$\hat{a} = 0.0077 (\text{true } a = 0.01) \quad \hat{V} = 0.0881 (\text{true } V = 0.1).$$

The optimization steps are shown in Fig.10.

Fig. 10. True and estimated states at $x = 0.9$

Fig. 11. The estimated MLE by using GA
VI. Conclusions

We formulated the stochastic distributed parameter systems without boundary conditions by using the boundary observation data. The Kalman filter is derived for the systems state and the boundary noise processes. From this the likelihood function is explicitly obtained and the consistency property of MLE is studied for the case that the number of observation mechanism becomes large. Some simulation results are presented for supporting the feasibility of the proposed scheme.

VII. Appendix-A

A. Derivation of (9)

For \( \phi \in H^2(0, 1) \) with \( \phi(0) = \phi(1) = 0 \), i.e., \( H_0^1 \cap H^2 \) and \( (\phi_1, \phi_2) = \int_0^1 \phi_1(x)\phi_2(x)dx \) (here we work in the usual Sobolev space \( H^m(0, 1) \) as used in Theorem-1.), (1) becomes

\[
(59) \quad d(u(t), \phi) = a(\frac{\partial^2 u}{\partial x^2}, \phi)dt + V(\frac{\partial u}{\partial x}, \phi)dt + (dw(t), \phi),
\]

\[
= a(u, \frac{\partial^2 \phi}{\partial x^2})dt - au(t, 1) \frac{\partial \phi(1)}{\partial x}dt + au(t, 0) \frac{\partial \phi(0)}{\partial x}dt
\]

\[
+ V(\frac{\partial u}{\partial x}, \phi)dt + (dw(t), \phi),
\]

(from (2) and (3))

\[
= a(u, \frac{\partial^2 \phi}{\partial x^2})dt + V(\frac{\partial u}{\partial x}, \phi)dt + a(dy_0(t) - \sigma dv^b_0(t)) \frac{\partial \phi(0)}{\partial x}
\]

\[
- a(dy_1(t) - \sigma_1 dv^b_1(t)) \frac{\partial \phi(1)}{\partial x} + (dw(t), \phi).
\]

Denoting that

\[
A = -a \frac{\partial^2 (-)}{\partial x^2} \text{ and } B = -V \frac{\partial (-)}{\partial x},
\]

(59) becomes

\[
(u(t), \phi) + \int_0^t (u(s), (A + B^*)\phi)ds
\]

\[
+a\{(y_1^b(t) - \sigma_1 v_1^b(t))(\frac{\partial \phi(1)}{\partial x}) - (y_0^b(t) - \sigma_0 v_0^b(t))(\frac{\partial \phi(0)}{\partial x})\}
\]

\[
= (u_0, \phi) + (u(t), \phi) \text{ for all } \phi \in H_0^1(0, 1) \cap H^2(0, 1),
\]

where \( B^* = V \frac{\partial (-)}{\partial x} \) and \( A = A^* \). Now for \( \phi \in H_0^1 \cap H^2 \), noting that \( \phi(0) = \phi(1) = 0 \) and integrating by parts with respect to \( x \), we have

\[
\left( \frac{\partial^2 \phi}{\partial x^2}, x(y_1^b(t) - \sigma_1 v_1^b(t)) + (1 - x)(y_0^b(t) - \sigma_0 v_0^b(t)) \right)
\]

\[
= (y_1^b(t) - \sigma_1 v_1^b(t)) \frac{\partial \phi(1)}{\partial x} - (y_0^b(t) - \sigma_0 v_0^b(t)) \frac{\partial \phi(0)}{\partial x}
\]

\[
- \int_0^1 \frac{\partial \phi}{\partial x} dx \{(y_1^b(t) - \sigma_1 v_1^b(t)) - (y_0^b(t) - \sigma_0 v_0^b(t))\}
\]

\[
= (y_1^b(t) - \sigma_1 v_1^b(t)) \frac{\partial \phi(1)}{\partial x} - (y_0^b(t) - \sigma_0 v_0^b(t)) \frac{\partial \phi(0)}{\partial x}.
\]
Hence we can include the boundary inputs into the interior system. Now (60) becomes

$$(61) \quad (u(t), \phi) + a(x(y_1^b(t) - \sigma_1 v_1^b(t)) + (1 - x)(y_0^b(t) - \sigma_0 v_0^b(t))), \frac{\partial^2 \phi}{\partial x^2} \biggr)$$

$$+ \int_0^t (u(s), (A + B^*)\phi) \, ds = (u_0, \phi) + (w(t), \phi).$$

Now our next task is to transform this system to the robust one. In (61) choosing $\phi = e_i(x) = \sqrt{2} \sin(i\pi x)$ and dividing this by $ai^2\pi^2$, we have

$$(62) \quad \frac{1}{ai^2\pi^2}(u(t), e_i(x)) - (x(y_1^b(t) - \sigma_1 v_1^b(t)) + (1 - x)(y_0^b(t) - \sigma_0 v_0^b(t))), e_i(x))$$

$$+ \int_0^t (u(s), (1 + \frac{1}{ai^2\pi}B^*)e_i(x)) \, ds = \frac{1}{ai^2\pi^2}(u_0, e_i(x)) + \frac{1}{ai^2\pi^2}(w(t, e_i(x), e_i(x)),$$

where we use $-a\frac{\partial^2 e_i(x)}{\partial x^2} = ai^2\pi^2 e_i(x)$. Now multiplying $e_i(x)$ to (62) and summing this up from $i = 1$ to $\infty$, we obtain

$$\sum_{i=1}^{\infty} \frac{1}{ai^2\pi^2}(u(t), e_i(x))e_i(x) - \sum_{i=1}^{\infty} \sum_{i=1}^{\infty} (x(y_1^b(t) - \sigma_1 v_1^b(t)) + (1 - x)(y_0^b(t) - \sigma_0 v_0^b(t))), e_i(x))$$

$$+ \int_0^t (u(s), \sum_{i=1}^{\infty} (1 + \frac{1}{ai^2\pi}B^*)e_i(x)) \, ds = \sum_{i=1}^{\infty} \frac{1}{ai^2\pi^2}(u_0, e_i(x))e_i(x)$$

$$+ \sum_{i=1}^{\infty} \frac{1}{ai^2\pi^2}(w(t, e_i(x))e_i(x),$$

Noting that the inverse of $A$ is given by

$$A^{-1} = \sum_{i=1}^{\infty} \frac{1}{a(i\pi)^2} \sqrt{2} \sin(i\pi x)(\sqrt{2} \sin(i\pi x), \cdot),$$

we get

$$(63) \quad A^{-1}u(t, x) - x(y_1^b(t) - \sigma_1 v_1^b(t)) - (1 - x)(y_0^b(t) - \sigma_0 v_0^b(t))$$

$$+ \int_0^t (1 + BA^{-1})u(s, x) \, ds = A^{-1}u_0(x) + A^{-1}w(t, x),$$

where we used the following relation:

$$\sum_{i=1}^{\infty} (\phi, \frac{1}{ai^2\pi^2}B^*e_i(x)) = \sum_{i=1}^{\infty} (B\phi, \frac{1}{ai^2\pi^2}e_i(x)) = A^{-1}B\phi = BA^{-1}\phi.$$

Hence, from (8) and $u = AA^{-1}u$, (63) becomes (9).

VIII. APPENDIX-B

A. Proof of Theorem-1

Noting that $A_\rho \phi = ae(-\frac{\mathcal{V}}{\partial x}) \{ \frac{\partial}{\partial x} \{ e \frac{\mathcal{V}}{\partial x} \frac{\partial \phi}{\partial x} \} \} = a\frac{\partial^2 \phi}{\partial x^2} + V \frac{\partial \phi}{\partial x}$, $A_\rho$ becomes a symmetric operator with the inner product $(\phi_1, ae\frac{\mathcal{V}}{\partial x} \phi_2)$. Hence, in this proof, for simplicity, we set $B = 0$. We use the usual Galerkin approximation method used by Pardoux [12]. Set $e_i(x) = \sqrt{2} \sin(i\pi x)$. The m-dimensional system related to (11) becomes for $i = 1, 2, 3, \cdots, m$

$$(\tilde{u}^m(t), e_i) + \int_0^t (\tilde{u}^m(s) + f(s, y), Ae_i) \, ds = (\tilde{u}_0, e_i) + (\tilde{u}^m(t), e_i),$$

S. AIHARA AND A. BAGCHI 19
where
\[ f(t, y) = x(\gamma_1^t(t) - \sigma_1 v_1^b(t)) + (1 - x)(\gamma_0^t(t) - \sigma_0 v_0^b(t)). \]

By using Ito’s lemma to \(|(\tilde{u}^m(t), e_i)|^2\), we have
\[ d|(\tilde{u}^m(t), e_i)|^2 + 2(\tilde{u}^m(t) + f(t, y), Ae_i)(\tilde{u}^m(t), e_i)dt \]
\[ = 2(\tilde{u}^m(t), e_i)(e_i, d\tilde{w}^m(t)) + (\tilde{Q}^m e_i, e_i)dt, \]
where \(\tilde{Q}^m = \sum_{i,j=1}^m (e_i, \tilde{Q} e_j) e_i \otimes e_j \). Noting that
\[ (\tilde{u}^m(t), Ae_i)(\tilde{u}^m(t), e_i) = a(i\pi)^2(\tilde{u}^m(t), e_i)^2, \]
we have
\[ |(\tilde{u}^m(t), e_i)|^2 + \int_0^t e^{-a(i\pi)^2(t-s)} a(i\pi)^2(\tilde{u}^m(t), e_i)^2 dt \]
\[ + \int_0^t e^{-a(i\pi)^2(t-s)} \{2(f(s, y), Ae_i)(\tilde{u}^m(t), e_i) \]
\[ = e^{-a(i\pi)^2t}(\tilde{u}^m_0, e_i)^2 + (\tilde{Q}^m e_i, e_i) \int_0^t e^{-a(i\pi)^2(t-s)} ds + 2 \int_0^t e^{-a(i\pi)^2(t-s)} (\tilde{u}^m(t), e_i)(e_i, d\tilde{w}^m(s)). \]

It is possible to derive the following estimate: for any \(\epsilon > 0, \exists C_1(\epsilon) > 0\)
\[ |(f(t, y), Ae_i)(\tilde{u}^m(t), e_i)| = a|f(t, y), i\pi e_i |i\pi(\tilde{u}^m(t), e_i)| \]
\[ = a\{ - (y_1^t(t) - \sigma_1 v_1^b(t)) + (y_0^b(t) - \sigma_1 v_0^b(t)) \} \sqrt{2(-1)} i\pi(\tilde{u}^m(t), e_i)| \]
\[ \leq \sqrt{2} a i\pi |(\tilde{u}^m(t), e_i)| |(y_1^t(t)| + |y_0^b(t)| + \sigma_1 |v_1^b| + \sigma_0 |v_0^b|) | \]
\[ \leq \frac{\epsilon}{2} a(i\pi)^2 |(\tilde{u}^m(t), e_i)|^2 + C_1(\epsilon) |(y_1^t(t)|^2 + |y_0^b(t)|^2 + \sigma_1^2 |v_1^b|^2 + \sigma_0^2 |v_0^b|^2). \]

Hence
\[ |(\tilde{u}^m(t), e_i)|^2 + a(i\pi)^2(1 - \frac{\epsilon}{2}) \int_0^t e^{-a(i\pi)^2(t-s)}(\tilde{u}^m(t), e_i)^2 dt \]
\[ \leq e^{-a(i\pi)^2t}(\tilde{u}^m_0, e_i)^2 + (\tilde{Q}^m e_i, e_i) \frac{1}{a i^2 \pi^2} (1 - e^{-a i^2 \pi^2 t}) + \int_0^t e^{-a(i\pi)^2(t-s)} g(s, y, v) ds \]
\[ + 2 \int_0^t e^{-a(i\pi)^2(t-s)} (\tilde{u}^m(t), e_i)(e_i, d\tilde{w}^m(s)), \]
where
\[ g(t, y, v) = C_1(\epsilon) |(y_1^t(t)|^2 + |y_0^b(t)|^2 + \sigma_1^2 |v_1^b|^2 + \sigma_0^2 |v_0^b|^2). \]

We find that from (A-1)
\[ E\{ \sup_t \int_0^t g(s, y, v) ds \} \leq C_4, \]
i.e.,
\[ E\{ \int_0^t e^{-a(i\pi)^2(t-s)} g(s, y, v) ds \} \leq C_4 \sum_{i=1}^m \frac{1}{a i^2 \pi^2} (1 - e^{-a i^2 \pi^2 t}) \]
and from (A-2) and (A-3)
\[ E\{ \sup_t \sum_{i=1}^m e^{-a(i\pi)^2t}|(\tilde{u}^m_0, e_i)|^2 \} + (Tr\{\tilde{Q}^m \} + C_4) \sum_{i=1}^m \frac{1}{a i^2 \pi^2} (1 - e^{-a i^2 \pi^2 t}) \leq C_5, \]
\[ \phi_1 \otimes \phi_2 = \phi_1(x)(\phi_2,) \]
for some constants $C_4$ and $C_5$ which are independent of $m$.

Choosing $\epsilon$ as

$$\alpha = 1 - \frac{\epsilon}{2} > 0,$$

we obtain

$$E\{\sum_{i=1}^{m} |(\tilde{u}^m(t), e_i)|^2\} + a\alpha E\{\sum_{i=1}^{m} t^2 \pi^2 \int_{0}^{t} e^{-a\pi^2(t-s)} |(\tilde{u}^m(s), e_i)|^2 ds\} \leq C_6.$$  

By using the Gronwall inequality, we have

$$\sup_{t} E\{|\tilde{u}^m(t)|^2\} \leq \text{Constant independent of } m,$$

and

$$E\{\sum_{i=1}^{m} \int_{0}^{t} e^{-a\pi^2(t-s)} t^2 \pi^2 |(\tilde{u}^m(s), e_i)|^2 ds\} \leq \text{Constant independent of } m.$$  

Consequently we can extract a subsequence of $\tilde{u}^m$ such that

$$\tilde{u}^m' \to \tilde{u} \text{ weakly star } L^\infty(T; L^2(\Omega; H)).$$

With the aid of the Burkholder-Davis-Gundy inequality, we also have

$$E\{\sup_{t} |\tilde{u}^m(t)|^2\} \leq \text{C}(E\{\sum_{i=1}^{m} \int_{0}^{t} (\tilde{u}^m(s), e_i^2) (e_i, \tilde{Q}e_i) ds\})^{1/2}.$$  

Hence we get

$$E\{\sup_{t} |\tilde{u}^m(t)|^2\} \leq \text{Const. independent of } m.$$

This implies that

$$\tilde{u}^m' \to \text{weakly star in } L^2(\Omega : L^\infty(T; H)).$$

\textbf{B. The proof of Theorem-2}

From (A-1)’ (using (2) and (3)), (11) becomes

$$\langle \tilde{u}(t), \tilde{\phi} \rangle + \int_{0}^{t} \langle \tilde{u}(s), x \int_{0}^{s} u(1, \tau) d\tau + (1 - x) \int_{0}^{s} u(0, \tau) d\tau, (A + B^*)\tilde{\phi} \rangle ds$$

$$= \langle \tilde{u}_0, \tilde{\phi} \rangle + \langle \tilde{w}(t), \tilde{\phi} \rangle.$$  

Defining

$$\tilde{u}(t) = \tilde{u}(t) + x \int_{0}^{t} u(1, \tau) d\tau + (1 - x) \int_{0}^{t} u(0, \tau) d\tau,$$

we get

$$\langle \tilde{u}(t), \tilde{\phi} \rangle + \int_{0}^{t} \langle \tilde{u}(s), (A + B^*)\tilde{\phi} \rangle ds = \langle \tilde{u}_0, \tilde{\phi} \rangle + \int_{0}^{t} \langle xu(1, s) + (1 - x)u(0, s), \tilde{\phi} \rangle ds$$

$$+ \langle \tilde{w}(t), \tilde{\phi} \rangle.$$  

Noting that

$$\tilde{u}(t, x) = 0, \text{ on } x = 0, 1,$$
(66) becomes

\[(\hat{u}(t), \hat{\phi}) + \int_0^t <(A + B)\hat{u}(s), \hat{\phi}> ds = (\hat{u}_o, \hat{\phi}) \]

\[+ \int_0^t (xu(1, s) + (1 - x)u(0, s), \hat{\phi}) ds + (\hat{w}(t), \hat{\phi}), \]

where \( <\cdot, \cdot > \) denotes the duality between \( V \) and \( V' \). Hence from (A-1)', it is a direct consequence from Pardoux [12] that

\[\hat{u} \in L^2(\Omega; C([0, T]; H)) \cap L^2([0, T]; V)\]

and then \( \hat{u} \) satisfies Theorem-2.

IX. APPENDIX-C

A. Proof of Proposition 3.1

1) \( p_{kk}, p_{23} \)-equations: Noting that from (39)

\[0 \leq p_{kk}(t; \theta) + \frac{1}{\sigma^2} \int_0^t |H\tilde{p}_{k1}(s, x; \theta)|^2 ds = \frac{t}{M}, \]

we have from \( \frac{t_f}{M} = C_f \)

\[(69) \quad \sup_{t \in T} |p_{kk}(t; \theta)| \leq \frac{C_f}{t_f}, \]

and

\[\int_0^{t_f} |H\tilde{p}_{k1}(s, x; \theta)|^2 ds \leq \sigma^2 \frac{t_f}{M} = \sigma^2 \frac{C_f}{t_f} \]

Hence it follows from (40) that \( ^5 \)

\[(70) \quad \sup_{t \in T} |p_{23}(t; \theta)| \leq \frac{1}{\sigma^2} \sqrt{\int_0^{t_f} |H\tilde{p}_{21}(s; \theta)|^2 ds \int_0^t |H\tilde{p}_{31}(s; \theta)|^2 ds} \]

\[\leq \frac{t_f}{M} \frac{C_f}{t_f} \]

2) \( \tilde{p}_{k1} \)-equations: It follows from (38) and \( \tilde{p}_{k1}(t; \theta) - x\sigma_1\tilde{p}_{k2}(t; \theta) - (1 - x)\sigma_0\tilde{p}_{k3}(t; \theta) = 0, \) on \( x = 0, 1 \) that

\[\frac{1}{2} \frac{d}{dt} |\tilde{p}_{k1}(t; \theta)|^2 + a |\frac{\partial \tilde{p}_{k1}(t; \theta)}{\partial x}|^2 + \delta |\tilde{p}_{k1}(t; \theta)|^2 \]

\[- \delta (x\sigma_1p_{k2}(t; \theta) + (1 - x)\sigma_0p_{k3}(t; \theta), \tilde{p}_{k1}(t; \theta)) \]

\[+ \frac{1}{\sigma^2} (\tilde{p}_{k1}(t; \theta), H^*H\tilde{P}_{11}(t; \theta)\tilde{p}_{k1}(t, x; \theta)) \]

\[= (a\sigma_1p_{k2}(t; \theta) - a\sigma_0p_{k3}(t; \theta) - V\tilde{p}_{k1}(t; \theta) \]

\[+ V(x\sigma_1p_{k2}(t; \theta) + (1 - x)\sigma_0p_{k3}(t; \theta)), \frac{\partial \tilde{p}_{k1}(t; \theta)}{\partial x}) \] .

Noting that \( (\phi, H^*H\tilde{P}_{11}(t; \theta)\phi) \geq 0, \)

\(^5\nu^0_0 \) and \( \nu^1_1 \) are mutually independent and hence \( p_{23}(0; \theta) = 0. \)
and from (27) and (29)

\[
(\tilde{p}_k(t;\theta), \frac{\partial \tilde{p}_k(t;\theta)}{\partial x}) = \frac{1}{2} \{\sigma_1^2|p_k(t;\theta)|^2 - \sigma_0|p_k(t;\theta)|^2\},
\]

\[\forall \epsilon > 0, \exists C(\epsilon) > 0 :\]

\[
\frac{1}{2} \frac{d}{dt} |\tilde{p}_k(t;\theta)|^2 + (a_m - \epsilon)|\frac{\partial \tilde{p}_k(t;\theta)}{\partial x}|^2 + (\delta - \epsilon)|\tilde{p}_k(t;\theta)|^2 \leq C(\epsilon)(|p_k(t;\theta)|^2 + |p_k(t;\theta)|^2)
\]

(71)

(from (69) and (70))

Choosing as \(a_m - \epsilon > 0\) and \(\delta - \epsilon > 0\), we have

\[
\frac{1}{2} \frac{d}{dt} |\tilde{p}_k(t;\theta)|^2 + (\delta - \epsilon)|\tilde{p}_k(t;\theta)|^2 \leq C(\epsilon) \frac{C_f^2}{t_f^4}.
\]

(72)

Hence we obtain

\[
|\tilde{p}_k(t;\theta)|^2 \leq e^{-2(\delta - \epsilon)t} \int_0^t e^{2(\delta - \epsilon)s} ds 2C(\epsilon) \frac{C_f^2}{t_f^4}
\]

\[
\leq 2C(\epsilon) \frac{C_f^2}{t_f^4}
\]

(73)

It follows from (71) that

\[
(a_m - \epsilon) \int_0^t |\frac{\partial \tilde{p}_k(s;\theta)}{\partial x}|^2 ds \leq \frac{1}{2} \frac{C_f^2}{t_f^4}
\]

(74)

3) \(\tilde{P}_{11}\)-equation: It follows that

\[
\frac{1}{2} \frac{d}{dt}[\tilde{P}_{11}(t;\theta), \tilde{P}_{11}(t;\theta)]
\]

\[+ [A^*_{R}(\theta) + B^*_{R}(\theta)) \tilde{P}_{11}(t;\theta), \tilde{P}_{11}(t;\theta) - \sigma_1 x \otimes \tilde{p}_{31}(t;\theta) - \sigma_0(1 - x) \otimes \tilde{p}_{31}(t;\theta)]
\]

\[+ [\tilde{P}_{11}(t;\theta), H^*H \tilde{P}_{11}(t;\theta) \tilde{P}_{11}(t;\theta)] = [\tilde{Q}, \tilde{P}_{11}(t;\theta)],
\]

where \(\phi_1 \otimes \phi_2 = \phi_1(x)(\phi_2, \cdot)\). It is easy to show that

\[
\frac{1}{2} \frac{d}{dt} [\tilde{P}_{11}(t;\theta), \tilde{P}_{11}(t;\theta)]
\]

\[+ [a^i \frac{\partial \tilde{P}_{11}(t;\theta)}{\partial x}, \tilde{P}_{11}(t;\theta) - \sigma_1 x \otimes \tilde{p}_{31}(t;\theta) + \sigma_0(1 - x) \otimes \tilde{p}_{31}(t;\theta)]
\]

\[+ [B^*_{R}(\theta) \tilde{P}_{11}(t;\theta), \tilde{P}_{11}(t;\theta) - \sigma_1 x \otimes \tilde{p}_{31}(t;\theta) - \sigma_0(1 - x) \otimes \tilde{p}_{31}(t;\theta)]
\]

\[+ [\delta \tilde{P}_{11}(t;\theta)^2 + \tilde{P}_{11}(t;\theta), -\sigma_1 x \otimes \tilde{p}_{31}(t;\theta) - \sigma_0(1 - x) \otimes \tilde{p}_{31}(t;\theta)] = [\tilde{Q}, \tilde{P}_{11}(t;\theta)].
\]

By using the same approach in the above subsection, \(\forall \epsilon_1, \epsilon_2 > 0\)

\[
\frac{1}{2} \frac{d}{dt} [\tilde{P}_{11}(t;\theta)]^2 + (a - \epsilon_1)|\frac{\partial \tilde{P}_{11}(t;\theta)}{\partial x}|^2 + (\delta - \epsilon_2)|\tilde{P}_{11}(t;\theta)|^2
\]

\[
\leq C(\epsilon_1, \epsilon_2) \{|	ilde{p}_{31}(t;\theta)|^2 + |	ilde{p}_{31}(t;\theta)|^2 + (Tr\{\tilde{Q}\})^2\}.
\]

(75)

( from (73))

\[
\leq C_1(\frac{C_f^2}{t_f^4} + (Tr\{\tilde{Q}\})^2).
\]

(76)
Hence

\[
[P_{11}(t; \theta)]^2 \leq e^{-2(\delta-\epsilon_2)t} \int_0^t e^{2(\delta-\epsilon_2)s} C_1 \left( \frac{C_f^2}{t_f^4} + (Tr\{\tilde{Q}\})^2 \right) ds \\
\leq C_2 \left( \frac{C_f^2}{t_f^4} + 1 \right).
\]

**B. Proof of Proposition 3.2**

1) \(\nabla_0 \tilde{p}_{kk,23}\)-equation: In this section, we only show the case for \(\theta_1 = a\), because the \(\theta_2 = V\) case is similar to the \(\theta_1 = a\) case. From (39,40), we have for \(k = 2, 3\)

\[
\nabla_{\theta_1} \tilde{p}_{kk}(t; \theta) = -\int_0^t \frac{2}{\sigma^2} \left( \nabla_{\theta_1} \tilde{p}_{k1}(s, x; \theta), H^* H \tilde{p}_{k1}(s, x; \theta) \right) ds
\]

and

\[
\nabla_{\theta_1} \tilde{p}_{23}(t; \theta) = -\int_0^t \frac{2}{\sigma^2} \left\{ (\nabla_{\theta_1} \tilde{p}_{31}(s, x; \theta), H^* H \tilde{p}_{31}(s, x; \theta)) + (\nabla_{\theta_1} \tilde{p}_{21}(s, x; \theta), H^* H \tilde{p}_{21}(s, x; \theta)) \right\} ds.
\]

From (73), we obtain for \(k = 2, 3\)

\[
\sup_{t \in T} |\nabla_{\theta_1} \tilde{p}_{kk}(t; \theta)| \leq C \frac{C_f}{t_f} \sup_{s \in T} |\nabla_{\theta_1} \tilde{p}_{k1}(s, x; \theta)|
\]

and

\[
\sup_{t \in T} |\nabla_{\theta_1} \tilde{p}_{23}(t; \theta)| \leq C \frac{C_f}{t_f} \sup_{s \in T} \left\{ \left| \nabla_{\theta_1} \tilde{p}_{21}(s, x; \theta) \right| + \left| \nabla_{\theta_1} \tilde{p}_{31}(s, x; \theta) \right| \right\} ds
\]

\[
\leq \frac{C C_f}{2} \sup_{s \in [0, t]} \left| \nabla_{\theta_1} \tilde{p}_{21}(s, x; \theta) \right| + \sup_{s \in [0, t]} \left| \nabla_{\theta_1} \tilde{p}_{31}(s, x; \theta) \right|
\]

where \(C\) is a constant independent of \(M\) and \(\theta\).

2) \(\nabla_{\theta_1} \tilde{p}_{k1}\)-equation: It follows from (38) that for \(k = 2, 3\)

\[
\left( \frac{d}{dt} \nabla_{\theta_1} \tilde{p}_{k1}(t, x; \theta), \phi \right) + (\nabla_{\theta_1} \tilde{p}_{k1}(t, x; \theta) - x \sigma_1 \nabla_{\theta_1} \tilde{p}_{k2}(t; \theta) - (1 - x) \sigma_0 \nabla_{\theta_1} \tilde{p}_{k3}(t; \theta), (A_d + B^*) \phi \right)
\]

\[
+ \frac{1}{\sigma^2} (\nabla_{\theta_1} \tilde{p}_{k1}(t, x; \theta), H^* H \tilde{p}_{11}(t; \theta) \phi) = -\frac{1}{\sigma^2} (\tilde{p}_{k1}(t, x; \theta), H^* H \nabla_{\theta_1} \tilde{p}_{11}(t; \theta) \phi)
\]

\[-(\tilde{p}_{k1}(t, x; \theta) - x \sigma_1 \tilde{p}_{k2}(t; \theta) - (1 - x) \sigma_0 \tilde{p}_{k3}(t; \theta), \frac{\partial^2 \phi}{\partial x^2}).
\]
Hence \(^6\)

\[
\frac{1}{2} \frac{d}{dt} |\nabla_{\theta_1} \tilde{p}_k(t, x; \theta)|^2 + a_1 |\frac{\partial}{\partial x} (\nabla_{\theta_1} \tilde{p}_k(t, x; \theta))|^2 + \delta |\nabla_{\theta_1} \tilde{p}_k(t, x; \theta)|^2 \\
+ \frac{1}{\sigma^2} (\nabla_{\theta_1} \tilde{p}_k(t, x; \theta), H^* H \tilde{P}_{11}(t, \theta) \nabla_{\theta_1} \tilde{p}_k(t, x; \theta)) = a(\sigma_1 \nabla_{\theta_1} \tilde{p}_k(t, \theta) - \sigma_0 \nabla_{\theta_1} \tilde{p}_k(t, \theta))^2 \\
+ \delta (\nabla_{\theta_1} \tilde{p}_k(t, x; \theta), x_1 \nabla_{\theta_1} \tilde{p}_k(t, x; \theta) + (1 - x) \sigma_0 \nabla_{\theta_1} \tilde{p}_k(t, x; \theta)) \\
+ \frac{1}{2} V \{ \sigma_1^2 |\nabla_{\theta_1} \tilde{p}_k(t, x; \theta)|^2 - \sigma_0^2 |\nabla_{\theta_1} \tilde{p}_k(t, x; \theta)|^2 \} \\
- V (\sigma_1 \nabla_{\theta_1} \tilde{p}_k(t, \theta) - \sigma_0 \nabla_{\theta_1} \tilde{p}_k(t, \theta)) \int_0^1 \nabla_{\theta_1} \tilde{p}_k(t, \theta) dx \}
\]

By using the similar procedure to derive (38) \(, \forall \epsilon_1 > 0, \epsilon_2 > 0, \exists C_1(\epsilon_1), C_2(\epsilon_2) : \)

\[
\frac{1}{2} \frac{d}{dt} |\nabla_{\theta_1} \tilde{p}_k(t, x; \theta)|^2 + (a - \epsilon_1) |\frac{\partial}{\partial x} (\nabla_{\theta_1} \tilde{p}_k(t, x; \theta))|^2 + (\delta - \epsilon_2) |\nabla_{\theta_1} \tilde{p}_k(t, x; \theta)|^2 \\
\leq C_1(\epsilon_1) \left( |\frac{\partial}{\partial x} \tilde{p}_k(t, x; \theta)|^2 + |\tilde{p}_k(t, x; \theta)|^2 \right) \\
+ C_2(\epsilon_2) \left( |\nabla_{\theta_1} \tilde{p}_k(t, x; \theta)|^2 + |\nabla_{\theta_1} \tilde{p}_k(t, x; \theta)|^2 + |\tilde{p}_k(t, x; \theta)|^2 \right)\right) \right). 
\]

Hence from (69), (70), (73) and (74), we have sup\(_{t \in T} |\tilde{p}_k(t, x; \theta)|^2 \leq \frac{C^2 t_f}{T^2} \) and 

\[
\int_0^{t_f} \left\{ \left| \frac{\partial}{\partial x} \tilde{p}_k(s, x; \theta) \right|^2 ds + t_f (\sup \| \tilde{p}_k(s, \theta) \|^2 + \sup \| \tilde{p}_k(s, \theta) \|^2) \right\} \leq \frac{C^2 t_f^2}{T^2}. 
\]

Finally

\[
|\nabla_{\theta_1} \tilde{p}_k(t, x; \theta)|^2 \leq C \left\{ \frac{CC^2 t_f}{T_f} + \int_0^t \left( |\nabla_{\theta_1} \tilde{p}_k(s, x; \theta)|^2 + |\nabla_{\theta_1} \tilde{p}_k(t, x; \theta)|^2 \right) ds \right\} \\
+ \frac{C^2 t_f}{T_f} |\nabla_{\theta_1} \tilde{P}_{11}(s; \theta)\tilde{h}|^2 \right) ds 
\]

(from (79))

\[
\leq C_1 \left( \frac{C^2 t_f}{T_f} + \frac{C^2}{t_f} \sup \| \nabla_{\theta_1} \tilde{p}_{21}(s, x; \theta) \|^2 + \frac{C^2}{t_f} \sup \| \nabla_{\theta_1} \tilde{p}_{31}(s, x; \theta) \|^2 \right) \\
+ \frac{C^2 t_f}{T_f} \sup \| \nabla_{\theta_1} \tilde{P}_{11}(s; \theta)\tilde{h}| \right). 
\]

Hence for sufficiently large \( t_f \), we have \( 1 - 2C_1 \frac{C^2}{T_f} \geq \alpha > 0 \) and 

\[
|\nabla_{\theta_1} \tilde{p}_{21}(t, x; \theta)|^2 + |\nabla_{\theta_1} \tilde{p}_{31}(t, x; \theta)|^2 \leq \frac{C}{t_f} (1 + \sup \| \nabla_{\theta_1} \tilde{P}_{11}(t; \theta)\tilde{h}|^2) / \alpha 
\]

\(^6\)We use the following boundary condition:

\[
\nabla_{\theta_1} \tilde{p}_k(t; \theta) - x_1 \nabla_{\theta_1} \tilde{p}_k(t; \theta) - (1 - x) \sigma_0 \tilde{p}_k(t; \theta) = 0, \text{ on } x = 0, 1. 
\]
3) $\nabla_{\theta} \tilde{P}_{11}$-equation: From (37), we have

$$
\begin{align*}
\frac{d}{dt} (\nabla_{\theta} \tilde{P}_{11}(t; \theta) \phi_1, \phi_2) \\
+ ((A_\delta + B^*) \phi_1), (\nabla_{\theta} \tilde{P}_{11}(t; \theta) - \sigma_1 x \otimes \nabla_{\theta} \tilde{p}_{21}(t; \theta) - \sigma_0 (1 - x) \otimes \nabla_{\theta} \tilde{p}_{31}(t; \theta)) \phi_2 \\
+ (\phi_1, (\nabla_{\theta} \tilde{P}_{11}(t; \theta) - \sigma_1 x \otimes \nabla_{\theta} \tilde{p}_{21}(t; \theta) - \sigma_0 (1 - x) \otimes \nabla_{\theta} \tilde{p}_{31}(t; \theta)) (A_\delta + B^*) \phi_2) \\
+ \frac{1}{\sigma^2} (\phi_1, (\nabla_{\theta} \tilde{P}_{11}(t; \theta)) H^* H \tilde{P}_{11}(t; \theta) + \tilde{P}_{11}(s; \theta) H^* H \tilde{P}_{11}(t; \theta)) \phi_2) \\
= - \left( \frac{\partial^2 \phi_1}{\partial x^2}, (\tilde{P}_{11}(t; \theta) - \sigma_1 x \otimes \tilde{p}_{21}(t; \theta) - \sigma_0 (1 - x) \otimes \tilde{p}_{31}(t; \theta)) \phi_2 \right) \\
- (\phi_1, (\tilde{P}_{11}(t; \theta) - \sigma_1 x \otimes \tilde{p}_{21}(s; \theta) - \sigma_0 (1 - x) \otimes \tilde{p}_{31}(t; \theta)) \frac{\partial^2 \phi_2}{\partial x^2}).
\end{align*}
$$

It is easy to show that

$$
\begin{align*}
\frac{1}{2} \frac{d}{dt} |\nabla_{\theta} \tilde{P}_{11}(t; \theta) \tilde{h}|^2 + 2a(\frac{\partial}{\partial x} \nabla_{\theta} \tilde{P}_{11}(t; \theta) \tilde{h})^2 + 2\delta |\nabla_{\theta} \tilde{P}_{11}(t; \theta) \tilde{h}|^2 \\
+ 2a(\frac{\partial}{\partial x} \nabla_{\theta} \tilde{P}_{11}(t; \theta) \tilde{h}) + \sigma_1 \nabla_{\theta} \tilde{p}_{21}(t; \theta) \tilde{h} + \sigma_0 \nabla_{\theta} \tilde{p}_{31}(t; \theta) \tilde{h} \\
+ 2\delta (\nabla_{\theta} \tilde{P}_{11}(t; \theta) \tilde{h}), -\sigma_1 x \nabla_{\theta} \tilde{p}_{21}(t; \theta) \tilde{h} - \sigma_0 (1 - x) \nabla_{\theta} \tilde{p}_{31}(t; \theta) \tilde{h} \\
- 2V(\frac{\partial}{\partial x} \nabla_{\theta} \tilde{P}_{11}(t; \theta) \tilde{h}), -\sigma_1 x \nabla_{\theta} \tilde{p}_{21}(t; \theta) \tilde{h} - \sigma_0 (1 - x) \nabla_{\theta} \tilde{p}_{31}(t; \theta) \tilde{h} \\
+ \frac{2}{\sigma^2} (\nabla_{\theta} \tilde{P}_{11}(t; \theta) \tilde{h}, \tilde{P}_{11}(t; \theta) H^* H \nabla_{\theta} \tilde{P}_{11}(t; \theta) \tilde{h}) \\
= -2(\nabla_{\theta} \tilde{P}_{11}(t; \theta) \tilde{h}, (\tilde{P}_{11}(t; \theta) - \sigma_1 x \otimes \tilde{p}_{21}(t; \theta) - \sigma_0 (1 - x) \otimes \tilde{p}_{31}(t; \theta)) \frac{\partial^2 \phi_2}{\partial x^2}).
\end{align*}
$$

From (A-6), we have $|\frac{\partial^2 \phi_2}{\partial x^2}| < \infty$. Hence for some $\tilde{\delta} > 0$, we obtain

$$
\begin{align*}
\frac{d}{dt} |\nabla_{\theta} \tilde{P}_{11}(t; \theta) \tilde{h}|^2 + & \tilde{\delta} |\nabla_{\theta} \tilde{P}_{11}(t; \theta) \tilde{h}|^2 \\
\leq & \ C \{ [\tilde{P}_{11}(t; \theta)]^2 + |\tilde{p}_{21}(t; \theta)|^2 + |\tilde{p}_{31}(t; \theta)|^2 + |\nabla_{\theta} \tilde{p}_{21}(t; \theta)|^2 + |\nabla_{\theta} \tilde{p}_{31}(t; \theta)|^2 \} \\
\leq & \ C_1 \{ 1 + \frac{C_f^2}{t_f^4} + \frac{C_2}{t_f^2} (1 + \sup_{s \in T} |\nabla_{\theta} \tilde{P}_{11}(s; \theta) \tilde{h}|^2) \}.
\end{align*}
$$

Hence we obtain

$$
\begin{align*}
|\nabla_{\theta} \tilde{P}_{11}(t; \theta) \tilde{h}|^2 \leq & \ e^{-\tilde{\delta} t} \int_0^t e^{\tilde{\delta} s} ds C_1 \{ 1 + \frac{C_f^2}{t_f^4} + \frac{C_2}{t_f^2} (1 + \sup_{s \in T} |\nabla_{\theta} \tilde{P}_{11}(s; \theta) \tilde{h}|^2) \} \\
\leq & \ \frac{C_1}{\tilde{\delta}} \{ 1 + \frac{C_f^2}{t_f^4} + \frac{C_2}{t_f^2} (1 + \sup_{s \in T} |\nabla_{\theta} \tilde{P}_{11}(s; \theta) \tilde{h}|^2) \}.
\end{align*}
$$

Choosing $1 - \frac{C_1 C_2}{\tilde{\delta} t_f^4} > 0$, we have

$$
|\nabla_{\theta} \tilde{P}_{11}(t; \theta) \tilde{h}|^2 + |\nabla_{\theta} \tilde{p}_{21}(t, \theta)|^2 + |\nabla_{\theta} \tilde{p}_{31}(t, \theta)|^2 \\
\leq \ \text{Constant independent of } M, t_f \text{ and } \theta.
$$
A. Derivation of (46) in Proof of Theorem-2

Define
\[
\hat{u}_g(t; \theta) = \hat{u}(t; \theta) + x y_0^{BM}(t) + (1 - x) y_0^{BM}(t) - \sigma_1 x \hat{v}_1^{BM}(t; \theta) - \sigma_0 (1 - x) v_0^{BM}(t; \theta).
\]

Now using the many boundary observations (31) and (32), we obtain
\[
\begin{align*}
(82) & \quad d(\hat{u}_g(t; \theta^o), \phi) + (\hat{u}_g(t; \theta), (A(\theta^o) + B^*(\theta^o)) \phi) dt = \{(\phi, \tilde{P}_{11}(t; \theta^o) \hat{h}) \\
& \quad - (\sigma_1 x (\tilde{p}_{21}(t; \theta^o), \hat{h}) + (\sigma_0 (1 - x) (\tilde{p}_{31}(t; \theta^o), \hat{h}), \phi) \} dz(t; \theta^o) \frac{1}{\sigma^2} \\
& \quad + (x u(1, t) + (1 - x) u(0, t), \phi) dt + \frac{1}{M} \sum_{i=1}^{M} (x \sigma_1 \nu_{1i}(t) + (1 - x) \sigma_0 \nu_{0i}(t), \phi).
\end{align*}
\]

For this filter we can apply the results of Theorem-2. Hence by using Ito’s lemma, we have
\[
\begin{align*}
(83) & \quad \frac{1}{2} \frac{d}{dt} E\{[(\hat{u}_g(t; \theta^o)]^2 \} + a^0 E\{[\frac{\partial}{\partial x} \hat{u}_g(t; \theta^o)]^2 \} = E\{(x u(1, t) + (1 - x) u(0, t), \hat{u}_g(t; \theta^o)) \} \\
& \quad + \frac{1}{\sigma^2} |\tilde{P}_{11}(t; \theta^o) \hat{h} - (\sigma_1 x (\tilde{p}_{21}(t; \theta^o), \hat{h}) - (\sigma_0 (1 - x) (\tilde{p}_{31}(t; \theta^o), \hat{h}))|^2 + \frac{1}{3M} (\sigma_0^2 + \sigma_1^2).
\end{align*}
\]

Noting that \( \hat{u}_g(t; \theta^o) = 0 \) on the boundary \( x = 0, 1 \), we have \( |\frac{\partial}{\partial x} \hat{u}_g(t; \theta^o)|^2 \geq \pi^2 |\hat{u}_g(t; \theta^o)|^2 \). Form Proposition 1 i.e., \( |\tilde{P}_{11}(t; \theta^o) \hat{h} - (\sigma_1 x (\tilde{p}_{21}(t; \theta^o), \hat{h}) - (\sigma_0 (1 - x) (\tilde{p}_{31}(t; \theta^o), \hat{h}))|^2 \leq C_1 \), and
\[
\frac{1}{3M} (\sigma_0^2 + \sigma_1^2) + C_1 \leq C_2,
\]

we have for \( \forall \epsilon > 0 \)
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} E\{[(\hat{u}_g(t; \theta^o)]^2 \} & + (a^0 \pi^2 - \epsilon) E\{[\hat{u}_g(t; \theta^o)]^2 \} \\
& \leq C(\epsilon) E\{|u(1, t)|^2 \} + E\{|u(0, t)|^2 \} + C_2.
\end{align*}
\]

Hence
\[
E\{[\hat{u}_g(t; \theta^o)]^2 \} \leq e^{-(2a^0 \pi^2 - \epsilon)t} C_2 + C(\epsilon) e^{-(2a^0 \pi^2 - \epsilon)t} \int_0^t (E\{|u(1, t)|^2 \} + E\{|u(0, t)|^2 \}) dt.
\]

It follows from (A-5) that
\[
(83) \quad E\{[\hat{u}_g(t; \theta^o)]^2 \} \leq \text{Constant independent of } t \text{ and } \theta.
\]

Define
\[
e(t; \theta, \theta^o) = \hat{u}_g(t; \theta) - \hat{u}_g(t; \theta^o).
\]

From \( \frac{\partial^2}{\partial x^2} (\hat{u}_g(t; \theta) - \hat{u}_g(t; \theta^o)) = \frac{\partial^2}{\partial x^2} (\hat{u}(t; \theta) - \hat{u}_g(t; \theta^o)) \) and (A-4), we have7
\[
H \phi = - \int_0^1 h \frac{\partial^2}{\partial x^2} \phi dx = \int_0^1 \frac{\partial h}{\partial x} \frac{\partial \phi}{\partial x} dx = - \int_0^1 \frac{\partial^2 h}{\partial x^2} \phi dx = \int_0^1 \hat{h}(x) \phi dx.
\]

It follows from (42), (43) and (82) that
\[
\begin{align*}
& \quad d(e(t; \theta, \theta^o), \phi) + (e(t; \theta, \theta^o), (A(\theta + B^*(\theta)) \phi) dt + (\hat{u}_g(t; \theta), (A(\theta - \theta^o) + B^*(\theta - \theta^o)) \phi) dt \\
& = \left\{ (\phi, \{\tilde{P}_{11}(t; \theta) - \tilde{P}_{11}(t; \theta^o)\} \hat{h}) - (\phi, \sigma_1 x (\tilde{p}_{21}(t; \theta) - \tilde{p}_{21}(t; \theta^o), \hat{h}) \\
& \quad - (\phi, \sigma_0 (1 - x) (\tilde{p}_{31}(t; \theta) - \tilde{p}_{31}(t; \theta^o), \hat{h}) \right\} dz(t; \theta^o) / \sigma^2 \\
& \quad + \left\{ - (\phi, \tilde{P}_{11}(t; \theta) \hat{h}) + (\phi, \sigma_1 x (\tilde{p}_{21}(t; \theta), \hat{h}) + (\phi, \sigma_0 (1 - x) (\tilde{p}_{31}(t; \theta), \hat{h})) \right\} H e(t; \theta, \theta^o) dt / \sigma^2.
\end{align*}
\]

7As stated in Sec.4, \( \tilde{h} \) depends on \( a \). Here for simplicity we set that \( \tilde{h} \) is independent of \( a \)without a loss of generality.
Introducing

\[ \Delta^{-1/2} = \sum_{i=1}^{\infty} \frac{1}{\pi i} \sqrt{2} \sin(i\pi x)(\sqrt{2} \sin(i\pi x), \cdot), \]

and

\[ \tilde{e}(t; \theta, \theta^o) = \Delta^{-1/2} e(t; \theta, \theta^o), \]

we have

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} E \{ |\tilde{e}(t; \theta, \theta^o)|^2 \} + a E \{ |\frac{\partial}{\partial x} \tilde{e}(t; \theta, \theta^o)|^2 \}
\end{align*}
\]

\[
\begin{align*}
+ E \{ (\Delta^{-1/2} \tilde{\gamma}_g(t; \theta)(A(\theta - \theta^o) + B^*(\theta - \theta^o)) \tilde{e}(t; \theta, \theta^o)) \}
\end{align*}
\]

\[
\begin{align*}
= |\Delta^{-1/2} \left\{ \{ \tilde{P}_{11}(t; \theta) - \tilde{P}_{11}(t; \theta^o) \} \tilde{h} - \sigma_1 x(\{ \tilde{p}_{21}(t; \theta) - \tilde{p}_{21}(t; \theta^o) \}, \tilde{h}) 
\end{align*}
\]

\[
\begin{align*}
- \sigma_0 (1 - x)(\{ \tilde{p}_{31}(t; \theta) - \tilde{p}_{31}(t; \theta^o) \}, \tilde{h}) \right\} |^2 \frac{1}{\sigma^2}
\end{align*}
\]

\[
\begin{align*}
- E \{ (\tilde{e}(t; \theta, \theta^o), \Delta^{-1/2} \tilde{P}_{11}(t; \theta) H^* H \Delta^{1/2} \tilde{e}(t; \theta, \theta^o)) \} \frac{1}{\sigma^2}
\end{align*}
\]

\[
\begin{align*}
+ E \{ (\tilde{e}(t; \theta, \theta^o), \Delta^{-1/2} \left\{ \sigma_1 x(\{ \tilde{p}_{21}(t; \theta), \tilde{h} \} + \sigma_0 (1 - x)(\{ \tilde{p}_{31}(t; \theta), \tilde{h} \}) \right\} H \Delta^{1/2} \tilde{e}(t; \theta, \theta^o)) \}
\end{align*}
\]

From Propositions 1 and 2, we get

\[
|\Delta^{-1/2} \left\{ \{ \tilde{P}_{11}(t; \theta) - \tilde{P}_{11}(t; \theta^o) \} \tilde{h} - \sigma_1 x(\{ \tilde{p}_{21}(t; \theta) - \tilde{p}_{21}(t; \theta^o) \}, \tilde{h}) 
\end{align*}
\]

\[
\begin{align*}
- \sigma_0 (1 - x)(\{ \tilde{p}_{31}(t; \theta) - \tilde{p}_{31}(t; \theta^o) \}, \tilde{h}) \right\} |^2 \frac{1}{\sigma^2} \leq C_3 |\theta - \theta^o|^2,
\end{align*}
\]

and

\[
|\Delta^{-1/2} \left\{ \sigma_1 x(\{ \tilde{p}_{21}(t; \theta), \tilde{h} \} + \sigma_0 (1 - x)(\{ \tilde{p}_{31}(t; \theta), \tilde{h} \}) \right\} H \Delta^{1/2} |_{HS} \leq C_{Cf} \frac{C_4}{t_f^2} |\Delta^{1/2} H|_{HS} \]

Hence \( \forall \epsilon_1, \epsilon_2 > 0, \exists C_1(\epsilon_1), C_2(\epsilon_2) > 0; \)

\[
\frac{1}{2} \frac{d}{dt} E \{ |\tilde{e}(t; \theta, \theta^o)|^2 \} + (a_m - \epsilon_1 - \epsilon_2) E \{ |\frac{\partial}{\partial x} \tilde{e}(t; \theta, \theta^o)|^2 \}
\]

\[
\leq C_1(\epsilon_1) E \{ |\frac{\partial}{\partial x} \Delta^{-1/2} \tilde{\gamma}_g(t; \theta)|^2 \} |a - a^o|^2 + C(\epsilon_2) E \{ |\Delta^{-1/2} \tilde{\gamma}_g(t; \theta)|^2 \}|V - V^o|^2
\]

\[
+ C_3 |\theta - \theta^o|^2 + C_{Cf} \frac{C_4}{t_f^2} E \{ |\tilde{e}(t; \theta, \theta^o)|^2 \},
\]

where from Propositions 1 and 2 we used

\[
|\Delta^{-1/2} \left\{ \{ \tilde{P}_{11}(t; \theta) - \tilde{P}_{11}(t; \theta^o) \} \tilde{h} - \sigma_1 x(\{ \tilde{p}_{21}(t; \theta) - \tilde{p}_{21}(t; \theta^o) \}, \tilde{h}) 
\end{align*}
\]

\[
\begin{align*}
- \sigma_0 (1 - x)(\{ \tilde{p}_{31}(t; \theta) - \tilde{p}_{31}(t; \theta^o) \}, \tilde{h}) \right\} |^2 \frac{1}{\sigma^2} \leq C_3,
\end{align*}
\]

and

\[
|\Delta^{-1/2} \left\{ \sigma_1 x(\{ \tilde{p}_{21}(t; \theta), \tilde{h} \} + \sigma_0 (1 - x)(\{ \tilde{p}_{31}(t; \theta), \tilde{h} \}) \right\} H \Delta^{1/2} |_{HS} \leq C_{Cf} \frac{C_4}{t_f^2} |\Delta^{1/2} H|_{HS} \]

\[
\leq \frac{C_{Cf}}{t_f^2} |\Delta^{1/2} H|_{HS} \leq C_{Cf} \frac{C_4}{t_f^2}.
\]
Noting that \( \frac{\nu^2}{M} = C_f \), we can choose \( M \) such that
\[
\alpha = (a_m - \epsilon_1 - \epsilon_2)\pi^2 - C_4 \frac{C_f}{f^2} > 0.
\]

From (83), we have
\[
E\{|\frac{\partial}{\partial x} \Delta^{-1/2} \hat{u}_y(t; \theta)|^2\} + E\{|\Delta^{-1/2} \hat{u}_y(t; \theta)|^2\} \leq CE\{|\hat{u}_y(t; \theta)|^2\} \leq \text{Const.}
\]

Hence
\[
(85) \quad \frac{1}{2} d \frac{d}{dt} E\{|\hat{e}(t; \theta, \theta^o)|^2\} + \alpha E\{|\hat{e}(t; \theta, \theta^o)|^2\} \leq C_5 |\theta - \theta^o|^2.
\]

This implies that
\[
E\{|\hat{e}(t; \theta, \theta^o)|^2\} \leq C_5 e^{-2\alpha t} |\theta - \theta^o|^2 \leq C_5 |\theta - \theta^o|^2.
\]

Consequently
\[
E\{|H(\hat{u}(t; \theta) - \hat{u}(t; \theta^o)|^2\} \leq |H\Delta^{1/2}_{HS} E\{|\hat{e}(t, \theta, \theta^o)|^2\} \leq C_6 |\theta - \theta^o|^2.
\]

XI. APPENDIX-E

In this Appendix we use the following notations:
\[
\tilde{p}(t, x) = A^{-1}p(t, x), \quad \tilde{p}(t, x, y) = A^{-1}(A^{-1}p(t, x, y)^*),
\]

and
\[
(\phi, \psi)_{\Gamma} = \phi(1)\psi(1) - \phi(0)\psi(0).
\]

A. \( p_{22}(t) \) and \( p_{33}(t) \) equations

Noting that
\[
-a \frac{\partial^2 \tilde{p}(t, x)}{\partial x^2} = p(t, x),
\]
we obtain
\[
\frac{dp_{22}(t)}{dt} + \frac{1}{\sigma_m} (\int_{G_o} h(x)p_{21}(t, x)dx)^2 = 1,
\]
\[
\frac{dp_{33}(t)}{dt} + \frac{1}{\sigma_m} (\int_{G_o} h(x)p_{31}(t, x)dx)^2 = 1,
\]
\[
\frac{dp_{23}(t)}{dt} + \frac{1}{\sigma_m} \int_{G_o} h(x)p_{21}(t, x)dx \int_{G_o} h(x)p_{31}(t, x)dx = 0.
\]
B. $p_{21}(t, x)$ and $p_{31}(t, x)$ equations

In (15), we set $\phi = A\psi$ for $\psi \in H^4 \cap H^1_0 \cap \{\frac{\partial^4 \psi(0)}{\partial x^4} = 0, \frac{\partial^4 \psi(1)}{\partial x^4} = 0\}$. Integrating by parts with respect to $x$, (15) becomes

$$
(\frac{\partial \tilde{p}_{21}(t, x)}{\partial t} - a \frac{\partial \psi}{\partial x})\Gamma + (\frac{\partial p_{21}(t, x)}{\partial t}, \psi) + a(\frac{\partial \tilde{p}_{21}(t, x)}{\partial x} - \sigma_1 p_{22}(t) + \sigma_0 p_{23}(t), \frac{\partial}{\partial x}(A + B^*)\psi)
$$

$$
+ \frac{1}{\sigma_m^2} \int_{G_o} h(x)p_{21}(t, x)dx \left\{ \int_{G_o} \tilde{h}(y)\tilde{p}_{11}(t, x, y)dy, -a \frac{\partial \psi}{\partial x}\right\} \Gamma
$$

$$
+ a\left( \int_{G_o} \tilde{h}(y)\frac{\partial \tilde{p}_{11}(t, x, y)}{\partial x}dy, \frac{\partial \psi}{\partial x}\right) = 0.
$$

Noting that

$$
a(\frac{\partial \tilde{p}_{21}(t, x)}{\partial x} - \sigma_1 p_{22}(t) + \sigma_0 p_{23}(t), \frac{\partial}{\partial x}(A + B^*)\psi)
$$

$$
= -V(\frac{\partial \tilde{p}_{21}(t, x)}{\partial x} - \sigma_1 p_{22}(t) + \sigma_0 p_{23}(t), -a \frac{\partial \psi}{\partial x})\Gamma
$$

$$
+(p_{21}(t, x), -a \frac{\partial \psi}{\partial x})\Gamma + ((A + B)p_{21}(t, x), \psi)
$$

and

$$
a\left( \int_{G_o} \tilde{h}(y)\frac{\partial \tilde{p}_{11}(t, x, y)}{\partial x}dy, \frac{\partial \psi}{\partial x}\right)
$$

$$
= -a\left( \int_{G_o} \tilde{h}(y)\frac{\partial^2 \tilde{p}_{11}(t, x, y)}{\partial x^2}dy, \psi\right) = \left( \int_{G_o} h(y)p_{11}(t, y, x)dy, \psi\right),
$$

we find that $p_{21}(t, x)$ equation in $0 < x < 1$ is given by

$$
\frac{\partial p_{21}(t, x)}{\partial t} - (a \frac{\partial^2}{\partial x^2} + V \frac{\partial}{\partial x})p_{21}(t, x) + \frac{1}{\sigma_m^2} \int_{G_o} h(y)p_{21}(t, y)dy \int_{G_o} h(y)p_{11}(t, y, x)dy = 0
$$

and

$$
(\frac{\partial \tilde{p}_{21}(t, x)}{\partial t} - a \frac{\partial \psi}{\partial x})\Gamma + \frac{1}{\sigma_m^2} \int_{G_o} h(y)p_{21}(t, y)dy \int_{G_o} \tilde{h}(y)\tilde{p}_{11}(t, x, y)dy, -a \frac{\partial \psi}{\partial x})\Gamma
$$

$$
-V(\frac{\partial \tilde{p}_{21}(t, x)}{\partial x} - \sigma_1 p_{22}(t) + \sigma_0 p_{23}(t), -a \frac{\partial \psi}{\partial x})\Gamma + (p_{21}(t, x), -a \frac{\partial \psi}{\partial x})\Gamma = 0.
$$

On the boundary $x = 1$ we find that from (27)

$$
\frac{\partial \tilde{p}_{21}(t, 1)}{\partial t} = \sigma_1 \frac{dp_{22}(t)}{dt},
$$

and from (25)

$$
\frac{1}{\sigma_m^2} \int_{G_o} h(y)p_{21}(t, y)dy \int_{G_o} \tilde{h}(y)\tilde{p}_{11}(t, 1, y)dy = \frac{1}{\sigma_m^2} \int_{G_o} h(y)p_{21}(t, y)dy \sigma_1 \int_{G_o} \tilde{h}(y)\tilde{p}_{21}(t, y)dy
$$

$$
= \frac{\sigma_1}{\sigma_m} \int_{G_o} h(y)p_{21}(t, y)dy^2.
$$

Hence

$$\sigma_1 \frac{dp_{22}(t)}{dt} + \frac{\sigma_1}{\sigma_m} \int_{G_o} h(y)p_{21}(t, y)dy^2
$$

$$
-V(\frac{\partial \tilde{p}_{21}(t, 1)}{\partial x} - \sigma_1 p_{22}(t) + \sigma_0 p_{23}(t)) + p_{21}(t, 1) = 0.
$$
Plugging $p_{23}(t)$ equation into (86), we have

\[(87) \quad p_{21}(t, 1) = -\sigma_1 + V\left(\frac{\partial \tilde{p}_{21}(t, 1)}{\partial x} - \sigma_1 p_{22}(t) + \sigma_0 p_{23}(t)\right).\]

Repeating the same procedure mentioned above, we obtain on $x = 0$

\[(88) \quad p_{21}(t, 0) = V\left(\frac{\partial \tilde{p}_{21}(t, 0)}{\partial x} - \sigma_1 p_{22}(t) + \sigma_0 p_{23}(t)\right).\]

Noting that

\[\int_{0}^{1} -x a \frac{\partial^2 \tilde{p}_{21}(t, x)}{\partial x^2} \, dx = -a \frac{\partial \tilde{p}_{21}(t, 1)}{\partial x} + a \tilde{p}_{21}(t, 1) - a \tilde{p}_{21}(t, 0)\]

we have

\[\int_{0}^{1} x p_{21}(t, x) \, dx = -a \frac{\partial \tilde{p}_{21}(t, 1)}{\partial x} + a \tilde{p}_{21}(t, 1) - a \tilde{p}_{21}(t, 0)\]

\[= -a \frac{\partial \tilde{p}_{21}(t, 1)}{\partial x} + a \sigma_1 p_{22}(t) - a \sigma_0 p_{23}(t).\]

Hence

\[\frac{\partial \tilde{p}_{21}(t, 1)}{\partial x} = \sigma_1 p_{22}(t) - \sigma_0 p_{23}(t) - \frac{1}{a} \int_{0}^{1} x p_{21}(t, x) \, dx.\]

We also have

\[\frac{\partial \tilde{p}_{21}(t, 0)}{\partial x} = \sigma_1 p_{22}(t) - \sigma_0 p_{23}(t) + \frac{1}{a} \int_{0}^{1} (1 - x) p_{21}(t, x) \, dx.\]

Consequently, substituting above two equations into (87) and (88), the boundary conditions (54) can be derived.

C. $p_{11}(t, x, y)$ equation

In (20), we reset $\psi$ and $\phi$ as $A \psi$ and $A \phi$ respectively and also assume that $\psi, \phi \in H^4 \cap H_0^1 \cap \{\frac{\partial^4 f(0)}{\partial x^4} = 0, \frac{\partial^4 f(1)}{\partial x^4} = 0\}$. 

From boundary conditions (24) and (25), by using integrating by parts, it is easy to show that

\[\int_{0}^{1} \int_{0}^{1} A\psi(x) \frac{\partial \tilde{p}_{11}(t, x, y)}{\partial t} A\phi(y) \, dx \, dy = a^2 \{\sigma_1^2 p_{22}(t) \frac{\partial \psi(1)}{\partial x} \frac{\partial \phi(1)}{\partial y} - \sigma_0 \sigma_1 p_{23}(t) \frac{\partial \psi(0)}{\partial x} \frac{\partial \phi(0)}{\partial y} \}
\]

\[+ a \{\sigma_1^2 \int_{0}^{1} \frac{\partial p_{21}(t, x)}{\partial t} \psi(x) \, dx \frac{\partial \phi(1)}{\partial y} + \sigma_0 \int_{0}^{1} \frac{\partial p_{31}(t, x)}{\partial t} \psi(x) \, dx \frac{\partial \phi(0)}{\partial y}\}
\]

\[+ a \{-\sigma_1^2 \int_{0}^{1} \frac{\partial p_{21}(t, x)}{\partial t} \phi(x) \, dx \frac{\partial \psi(1)}{\partial y} + \sigma_0 \int_{0}^{1} \frac{\partial p_{31}(t, x)}{\partial t} \phi(x) \, dx \frac{\partial \psi(0)}{\partial y}\}
\]

\[+ \int_{0}^{1} \int_{0}^{1} p_{11}(t, x, y) \phi(y) \psi(y) \, dx \, dy,\]
\[
\int_0^1 \int_0^1 A \psi(x) \{ \bar{p}_{11}(t, x, y) - \sigma_1 \bar{p}_{21}(t, x) y - \sigma_0 \bar{p}_{31}(t, x)(1 - y) \} (A + B^*) A \phi(y) dy dx
\]

\[
= -a^2 \sigma_1^2 \frac{\partial \psi(1)}{\partial x} \frac{\partial \phi(1)}{\partial y} - a^2 \sigma_0^2 \frac{\partial \psi(0)}{\partial x} \frac{\partial \phi(0)}{\partial y}
\]

\[
- a \sigma_1 \int_0^1 \left( -a \frac{\partial^2}{\partial y^2} - V \frac{\partial}{\partial y} \right) p_{21}(t, y) \phi(y) dy \frac{\partial \psi(1)}{\partial x}
\]

\[
+ a \sigma_0 \int_0^1 \left( -a \frac{\partial^2}{\partial y^2} - V \frac{\partial}{\partial y} \right) p_{31}(t, y) \phi(y) dy \frac{\partial \psi(0)}{\partial x}
\]

where to derive above equation we used the following relation:

\[
\frac{\partial^3 \bar{p}_{11}(t, x, 1)}{\partial x^2 \partial y} = -\frac{1}{a} \sigma_1 p_{21}(t, x) + \frac{\sigma_0}{a} p_{31}(t, x) + \frac{1}{a^2} \int_0^1 \int_0^1 y p_{11}(t, x, y) dy
\]

\[
\frac{\partial^3 \bar{p}_{11}(t, x, 0)}{\partial x^2 \partial y} = -\frac{1}{a} \sigma_1 p_{21}(t, x) + \frac{\sigma_0}{a} p_{31}(t, x) - \frac{1}{a^2} \int_0^1 \int_0^1 y p_{11}(t, x, y) dy.
\]

We also have

\[
\int_0^1 \int_0^1 A \psi(x) \{ \bar{p}_{11}(t, x, y) - \sigma_1 \bar{p}_{21}(t, x) y - \sigma_0 \bar{p}_{31}(t, x)(1 - y) \} (A + B^*) A \phi(y) dy dx
\]

\[
= -a^2 \sigma_1^2 \frac{\partial \psi(1)}{\partial x} \frac{\partial \phi(1)}{\partial y} - a^2 \sigma_0^2 \frac{\partial \psi(0)}{\partial x} \frac{\partial \phi(0)}{\partial y}
\]

\[
- a \sigma_1 \int_0^1 \left( -a \frac{\partial^2}{\partial x^2} - V \frac{\partial}{\partial x} \right) p_{21}(t, x) \psi(x) dx \frac{\partial \phi(1)}{\partial y}
\]

\[
+ a \sigma_0 \int_0^1 \left( -a \frac{\partial^2}{\partial x^2} - V \frac{\partial}{\partial x} \right) p_{31}(t, x) \psi(x) dx \frac{\partial \phi(0)}{\partial y}
\]

+ \int_0^1 \left( -\bar{a} p_{11}(t, 1, y) \right) - V \int_0^1 x p_{11}(t, x, y) dy \psi(y) dy \frac{\partial \phi(1)}{\partial x}
\]

\[
+ \int_0^1 \left( \bar{a} p_{11}(t, 0, y) \right) - V \int_0^1 (1 - x) p_{11}(t, x, y) \psi(y) dy \phi(x) dx \frac{\partial \psi(0)}{\partial y}
\]

+ \int_0^1 \left( -a \frac{\partial^2}{\partial x^2} - V \frac{\partial}{\partial x} \right) p_{11}(t, x, y) \phi(x) \psi(y) dy dx.
The quadratic term becomes

$$\int_0^1 \int_{G_o} \tilde{h}(z) \tilde{p}_{11}(t,x,z)dz \quad A \phi(x) dx \int_0^1 \int_{G_o} \tilde{h}(z) \tilde{p}_{11}(t,y,z)dz A \psi(y) dy$$

$$= a^2 \sigma_1^2 \left( \int_{G_o} h(x)p_{21}(t,x)dx \right)^2 \frac{\partial \phi(0)}{\partial y} \frac{\partial \psi(1)}{\partial y}$$

$$- a^2 \sigma_1 \sigma_0 \int_{G_o} h(x)p_{31}(t,x)dx \int_{G_o} h(y)p_{21}(t,y)dy \frac{\partial \phi(0)}{\partial y} \frac{\partial \psi(1)}{\partial y}$$

$$- a^2 \sigma_1 \sigma_0 \int_{G_o} h(x)p_{31}(t,x)dx \int_{G_o} h(y)p_{21}(t,y)dy \frac{\partial \phi(1)}{\partial y} \frac{\partial \psi(0)}{\partial y}$$

$$+ a^2 \sigma_0^2 \left( \int_{G_o} h(x)p_{31}(t,x)dx \right)^2 \frac{\partial \phi(0)}{\partial y} \frac{\partial \psi(0)}{\partial y}$$

$$+ \sigma_1 \int_0^t \int_{G_o} h(x)p_{21}(t,x)dx \int_{G_o} h(z)p_{11}(t,y,z)dz \psi(y)dy (-a \frac{\partial \phi(1)}{\partial x})$$

$$- \sigma_1 \int_0^t \int_{G_o} h(x)p_{31}(t,x)dx \int_{G_o} h(z)p_{11}(t,y,z)dz \psi(y)dy (-a \frac{\partial \phi(0)}{\partial x})$$

$$+ \sigma_1 \int_0^t \int_{G_o} h(x)p_{21}(t,x)dx \int_{G_o} h(z)p_{11}(t,y,z)dz \phi(x)dx (-a \frac{\partial \psi(1)}{\partial x})$$

$$- \sigma_1 \int_0^t \int_{G_o} h(x)p_{31}(t,x)dx \int_{G_o} h(z)p_{11}(t,y,z)dz \phi(x)dx (-a \frac{\partial \psi(0)}{\partial x})$$

$$+ \int_0^1 \int_0^1 \int_{G_o} h(z)p_{11}(t,x,z)dz \int_{G_o} h(z)p_{11}(t,y,z)dz \psi(x) \psi(y)dx dy.$$ 

Hence summing up above equations and using $p_{21}, p_{31}, p_{22}$ and $p_{23}$ equations, (23) can be derived. For the boundary conditions, we have

$$a^2 \sigma_0^2 \frac{\partial \psi(0)}{\partial y} + a^2 \sigma_1^2 \frac{\partial \psi(1)}{\partial y} + \int_0^1 \{ -ap_{11}(t,x,1) - V \int_0^1 y p_{11}(t,x,y)dy \} \psi(x) dx \frac{\partial \phi(1)}{\partial y}$$

$$+ \int_0^1 \{ ap_{11}(t,x,0) - V \int_0^1 (1-y) p_{11}(t,x,y)dy \} \psi(x) dx \frac{\partial \phi(0)}{\partial y}$$

$$+ \int_0^1 \{ -ap_{11}(t,1,y) - V \int_0^1 x p_{11}(t,x,y)dx \} \phi(y) dy \frac{\partial \psi(1)}{\partial y}$$

$$+ \int_0^1 \{ ap_{11}(t,0,y) - V \int_0^1 (1-x) p_{11}(t,x,y)dx \} \phi(y) dy \frac{\partial \psi(0)}{\partial y} = 0.$$

Now introducing

$$\int_0^1 \frac{\partial}{\partial x} \delta(x) \phi(x) dx = \frac{\partial \phi(0)}{\partial x}, \quad \int_0^1 \frac{\partial}{\partial x} \delta(x-1) \phi(x) dx = \frac{\partial \phi(1)}{\partial x},$$

we can derive the boundary conditions (49).

REFERENCES


