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Optimal search: a practical interpretation of information-driven sensor management

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***Abstract*—We consider the problem of scheduling an agile sensor for performing optimal search for a target.**

A probability density function is created for representing our knowledge about where the target might be and it is utilized by the proposed sensor management criteria for finding optimal search strategies.

The proposed criteria are: an information-driven criterion based on the Kullback-Leibler divergence and a criterion with practical meaning, i.e. performing the sensing action that will yield the maximum probability of detecting the target.

It is shown that using the aforementioned criteria results in the same sensing actions when searching for a target and this result establishes a practical operational justification for using information-driven sensor management for performing search.

I. Introduction

The problem of performing search emerges when the available sensor resources have to be utilized in an efficient way such that the search for an object or a feature is successful.

The challenges are to find the object as soon as possible while spending as few resources as possible. Towards this goal, sensor management criteria can be utilized. The main advantage of using such criteria over the simple approaches of periodic or random search is that the criteria, if carefully chosen, can demonstrate adaptive behavior when external information is available. For instance, if the object is expected to be with higher probability in a specific region, the periodic or random search approaches would not take this information into account but a carefully chosen or designed criterion would produce search patterns that leverage this information in order to find the object faster and/or by using less resources. If the external information is updated at each iteration, like in our case, then the problem amounts to performing one-step ahead (or myopic) optimal search.

Some examples where these challenges appear are: target detection [1], [2], search for wreckages and survivors [3], [4], search for intruders etc. Especially the last example is closely related to the pursuit-evasion problems that have been studied under different assumptions and solved using different approaches in the robotics community [5], [6].

We consider the scheduling of an agile sensor for efficiently searching for a target. A characteristic example of such a sensor is a multifunction radar (MFR). Such a radar has received a lot of focus from the research community as an attempt to schedule efficiently its tasks, one of which is to perform search for undetected targets.

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In [7] the track and search functions of an MFR are scheduled according to a threat-based criterion. For scheduling search functions, the authors use ghost targets that dictate volume or horizon search instead of tracking radar functions.

In [8] the revisit intervals, radar beam positions, and energy per dwell are controlled for improving track quality and energy efficiency. Especially in the case of searching, the use of negative information is suggested for updating the predictive densities of the targets and obtaining a search pattern by searching the region where the maximum of the predictive density is located. An updated version is [9].

In [10] the authors use a *search-to-track* ratio that the user has to set. According to this ratio, the sensor manager schedules the corresponding tasks of the radar. When the search task is considered, an estimate of the spatial density of previously undetected targets is utilized. The sensing action that maximizes the expected number of newly detected targets is chosen whenever a search function is scheduled. A disadvantage of this approach is that the search-to-track ratio is user defined and not automatically determined by the scheduling algorithm according to the optimization of a criterion. A similar scheduling approach is presented in [11] where the scheduling criterion suggests selecting recursively those sensors that cover the most probability mass of the predictive density.

In [12] an approach similar to ours has been proposed. An a priori probability distribution of the target to be detected is specified by a set of discrete target position probabilities corresponding to each search beam. Immediately after the increment of search effort is applied, the target position probability density is updated by the use of Bayes' rule. The proposed solution suggests making the next look in the search cell that will provide the maximum value of the incremental search energy and S/N payoff ratios (target cumulative probability of detection increase divided by search effort expenditure increase) for all cells and to maximize the duty factor of each cell.

In [13] the authors introduce the continuous double auction parameter selection algorithm (CDAPS) which manages the MFR resources by utilizing an auction mechanism to select parameters for individual radar tasks. The authors show that their algorithm performs better than periodic search.

The approach presented in our paper builds on the approaches described in the literature and the specific contributions are:

- The construction of a probability density of the undetected target and its implementation using a particle filter.
- The implementation of two sensor management criteria based on the aforementioned density: a criterion based on Kullback-Leibler divergence and a criterion based on the expected probability of detection.
- It is proven that the two aforementioned criteria are equivalent, in the sense that they lead to the same sensor selection scheme, under certain conditions.

The importance of this result lies in the connection that is established between a criterion that is optimal in the information theory context but has no practical meaning, i.e. maximizing the expected Kullback-Leibler divergence, and a criterion that has straightforward practical meaning, i.e. choosing the action that will yield the maximum probability of detecting the target.

The rest of the paper is organized as follows. In section II the system description is given and the problem under consideration is described. In section III the proposed solution is presented and in section IV a graphical proof of equivalence of the proposed sensor management criteria is given. In section V the simulation results are presented. Finally, in section VI the conclusions are discussed along with some open questions.

II. System setup and problem formulation

Consider a scenario where an agile sensor has to search for one target. This system can be described mathematically by the following (discrete time) state and measurement equations:

$$s_k = f(s_{k-1}, w_{k-1}) \tag{1}$$

$$z_k = \begin{cases} \{\emptyset\}, & \text{no target present} \quad \text{(2a)} \\ h(s_k, u_k, v_k), & \text{one target present} \quad \text{(2b)} \end{cases}$$

$$s_0 \sim p(s_0) \tag{3}$$

where

- $k = 1, 2, \dots$ is the time index
- $s_k \in \mathbb{R}^{N_s}$ is the state of the system at time k
- $w_k \in \mathbb{R}^{N_s}$ is the process noise with probability density $p_w(w_k)$
- $u_k \in U$ is the chosen sensing action, with U being the set of the available sensing actions
- $z_k \in \mathbb{R}^{N_z}$ is the received measurement with dimensionality N_z. If there is no target, then there will be no measurement and therefore (2a) will hold.
- v_k is the N_z-dimensional measurement noise with probability density $p_v(v_k)$
- s_0 is the initial state of the system with probability density $p(s_0)$
- the vector and possibly non-linear function $f(\cdot) : \mathbb{R}^{N_s} \mapsto \mathbb{R}^{N_s}$ describes the dynamics of the system
- similarly, the vector and possibly non-linear function $h(\cdot) : \mathbb{R}^{N_s} \mapsto \mathbb{R}^{N_z}$ relates the measurement z_k to the system state s_k and the sensing action u_k

The considered problem amounts to finding the best sensing action u_k by maximizing a sensor management criterion $V(s_k, z_k, u)$

$$u_k = \arg\max_u V(s_k, z_k, u) \tag{4}$$

and then using it for solving the attached filtering problem of determining the posterior probability density function $p(s_k|Z_k, U_k)$ that describes where the target might be. We denote by $Z_k = \{z_1, ..., z_k\}$ the measurement history and by $U_k = \{u_1, ..., u_k\}$ the sensing action history.

III. Proposed solution

We propose solving the described problem by employing the recursive Bayesian estimation approach implemented by a particle filter and performing the optimization of the criteria using quantities of the running particle filter. The result will be a sensing action optimal in the context of the criteria.

A. Recursive Bayesian estimation

In the recursive Bayesian estimation context, given a probability density function $p(s_{k-1}|Z_{k-1}, U_{k-1})$, first the prediction step is performed using the Chapman-Kolmogorov equation:

$$p(s_k|Z_{k-1}, U_{k-1}) = \int p(s_k|s_{k-1}) p(s_{k-1}|Z_{k-1}, U_{k-1})\, ds_{k-1} \tag{5}$$

where $p(s_k|s_{k-1})$ is determined by the kinematic model of the target.

Then the predictive density $p(s_k|Z_{k-1}, U_{k-1})$ is updated with the received measurement z_k using Bayes' rule

$$p(s_k|Z_k, U_k) = \frac{p(z_k|s_k, u_k) \cdot p(s_k|Z_{k-1}, U_{k-1})}{p(z_k|Z_{k-1}, U_k)} \tag{6}$$

$$\propto p(z_k|s_k, u_k) \cdot p(s_k|Z_{k-1}, U_{k-1}) \tag{7}$$

where $p(z_k|s_k, u_k)$ is the likelihood function and

$$p(z_k|Z_{k-1}, U_k) = \int p(z_k|s_k, u_k) \cdot p(s_k|Z_{k-1}, U_{k-1})\, ds_k \tag{8}$$

is a normalizing constant which in practice does not have to be calculated if a particle filter is employed.

We will use a standard SIR particle filter [14] for approximating Equations (5) and (7) with N particles s_k^i and corresponding weights q_k^i:

$$\{s_k^i, q_k^i\},\ i = 1, ..., N \tag{9}$$

such that the approximation converges to the true posterior distribution $p(s_k|Z_k, U_k)$ as $N \to \infty$, see [15].

B. Dynamical model

The state of the system is assumed to be 4-dimensional, describing the position and velocity of the target in Cartesian coordinates

$$s_k = [x_k \quad v_x \quad y_k \quad v_y]^T \in \mathbb{R}^4 \tag{10}$$

The following target dynamics are also assumed:

$$s_k = f(s_{k-1}, w_{k-1}) = F \cdot s_{k-1} + w_k \tag{11}$$

where:

$$w_k \sim \mathcal{N}(\mu, \Sigma)$$

$$F = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} b_x T^3/3 & b_x T^2/2 & 0 & 0 \\ b_x T^2/2 & b_x T & 0 & 0 \\ 0 & 0 & b_y T^3/3 & b_y T^2/2 \\ 0 & 0 & b_y T^2/2 & b_y T \end{bmatrix}$$

and $b_x = b_y$ are the power spectral densities of the acceleration noise in the $x - y$ direction, T is the sampling time and $\mu = [0\ 0\ 0\ 0]^T$ is the mean of the Gaussian noise.

C. Measurement model and its use in the update step

The search for an undetected target is considered. This implies that no measurements are received or equivalently that the measurement z_k is *always* an empty set (Eq. 2a) and the measurement history is a vector of empty sets. Furthermore, we assume that no false alarms are present (but this assumption can be relaxed in a straightforward manner):

$$Z_k = \{\emptyset, \emptyset, \ldots\} \tag{12}$$

The aforementioned assumption means that the system operates in the context of *Negative Information* [9]. Therefore, if the probability of detecting the target when performing the sensing action u_k is defined as $P_d(s_k, u_k) \in (0, 1)$ then the likelihood function becomes

$$p(z_k|s_k, u_k) = p(z_k = \{\emptyset\}|s_k, u_k) = 1 - P_d(s_k, u_k) \tag{13}$$

From now on $z_k = \{\emptyset\}$ and $Z_k = \{\emptyset, \emptyset, \ldots\}$ will be skipped in the notation for simplicity reasons and we will only write $p(s_k|U_k)$ etc.

Given the aforementioned simplification, the prediction step in Eq. (5) becomes:

$$p(s_k|U_{k-1}) = \int p(s_k|s_{k-1}) \cdot p(s_{k-1}|U_{k-1})\ ds_{k-1} \tag{14}$$

and the update step in Eq. (6) becomes:

$$p(s_k|U_k) = \frac{[1 - P_d(s_k, u_k)] \cdot p(s_k|U_{k-1})}{\mathcal{C}} \tag{15}$$

$$\propto [1 - P_d(s_k, u_k)] \cdot p(s_k|U_{k-1}) \tag{16}$$

with

$$\mathcal{C} = \int [1 - P_d(s_k, u_k)] \cdot p(s_k|U_{k-1})\ ds_k \tag{17}$$

a normalizing constant that does not need to be calculated when a particle filter is employed.

D. Sensor management criteria

Our knowledge about the location of the undetected target is represented by a probability density function and consequently, the uncertainty about this knowledge (or the information gain by means of performing search) can be conveniently described in the information theory context.

We use the expected Kullback-Leibler divergence (KLD) in order to contribute to the ongoing discussion on whether task-based or information-driven criteria should be used in sensor management and what the practical interpretation of the latter is (a more elaborate discussion on this subject can be found in [16]). The maximum expected KLD will be compared to a practical (task-based) criterion that selects the search action that will yield the maximum expected probability of detecting the target.

In all the following formulas for the particle approximations it will hold that the weights of all the particles will be $q_k^i = 1/N$ because resampling is performed at every time step and that $s_k^i, s_k^j \sim p(s_k|U_{k-1})$.

1) Maximum expected Kullback-Leibler divergence: Maximizing the expected KL divergence between the posterior and the predictive density has been shown to lead to the same sensing actions as minimizing the conditional entropy or maximizing the mutual information under two conditions [16]. The two conditions for this claim to be valid are: the target should not adapt its motion strategy to our sensing strategy, and the ordering of the arguments in the evaluation of the KL divergence should be: $KL(q(s)||p(s))$ where $q(s)$ is the posterior density and $p(s)$ is the predictive density [16]. We choose to implement the maximum expected KL divergence because its computation is the least expensive, see the particle approximations in [17], [18].

The KL divergence between two densities $q(s)$ and $p(s)$ is given by

$$KL[\mathbf{q}||\mathbf{p}] = \int q(s) \cdot \log\left(\frac{q(s)}{p(s)}\right)\ ds \tag{18}$$

As suggested in [17] for example, the maximum expected KL divergence between the predictive and the simulated posterior density can be used for choosing the most informative sensing action u_k. The sensor management criterion would then be:

$$u_k = \arg\max_u \; E_{\mathcal{Z}}\left[KL(\mathbf{q}||\mathbf{p})\right] = \arg\max_u \; \left[KL(\mathbf{q}||\mathbf{p})\right] \quad (19)$$

where

$$\mathbf{q} = p(s_k|u, U_{k-1}) \quad (20)$$
$$\mathbf{p} = p(s_k|U_{k-1}) \quad (21)$$

The expectation over the measurement space $\mathcal{Z}$ is trivial and is not shown in Eq. (19) because of the assumption that the measurement will always be an empty set, see Eq. (2a).

If we set Eq. (20) equal to Eq. (15) and substitute the result and Eq. (21) in Eq. (18) then we obtain:

$$KL[\mathbf{q}||\mathbf{p}] = \int \frac{1 - P_d(s_k, u)}{\mathcal{C}} \cdot \log\left(\frac{1 - P_d(s_k, u)}{\mathcal{C}}\right) p(s_k|U_{k-1})\; ds_k \quad (22)$$

The particle approximation of Eq. (22) is given by:

$$KL[\mathbf{q}||\mathbf{p}] \approx \frac{1}{N}\sum_{i=1}^{N}\left\{\frac{1 - P_d(s_k^i, u)}{\hat{\mathcal{C}}} \cdot \log\left(\frac{1 - P_d(s_k^i, u)}{\hat{\mathcal{C}}}\right)\right\} \quad (23)$$

and

$$\mathcal{C} = \int [1 - P_d(s_k, u)] \cdot p(s_k|U_{k-1})\; ds \approx \frac{1}{N}\sum_{j=1}^{N}\left\{1 - P_d(s_k^j, u)\right\} = \hat{\mathcal{C}} \quad (24)$$

where $s_k^i \sim p(s_k|U_{k-1})$

2) Maximum expected probability of detection: Even though the criterion based on KL divergence is optimal in the information theory context, it is not easy to explain its practical meaning. For example, how could we describe its practical interpretation when we want to motivate our criterion choice to a radar operator? For this reason, the usage of criteria that have practical operational meaning is explored. The criterion chosen from this set of criteria suggests performing the sensing action that will yield the maximum expected probability of detecting the target. The choice of this specific criterion has been motivated by the works presented in [10], [11].

Given a probability density function $q(s)$ that describes where the target might be and the probability of detection function $P_d(s, u)$ that depends on the location of the target and the sensing action u, the probability of detecting the target if we perform the action u is given by:

$$\hat{P}_D = \int P_d(s, u) \cdot q(s)\; ds \quad (25)$$

In the considered scenario we use the predictive density $p(s_k|U_{k-1})$ in order to define a criterion that selects the sensing action u_k that will yield the maximum probability of detecting the target:

$$u_k = \arg\max_u \left[\int P_d(s_k, u) \cdot p(s_k|U_{k-1})\; ds_k\right] \quad (26)$$

The particle approximation of Eq. (26) is:

$$u_k = \arg\max_u \left[\int P_d(s_k, u) \cdot p(s_k|U_{k-1})\; ds_k\right] \approx \arg\max_u \left[\frac{1}{N}\sum_{i=1}^{N} P_d(s_k^i, u)\right] \quad (27)$$

where $s_k^i \sim p(s_k|U_{k-1})$

IV. Proof of equivalence of the two criteria

In the simplest case scenario, where the probability of detecting the target is constant, it can be proven that the two criteria are equivalent. The mathematical proof can be found at the Appendix and only a graphical explanation of the proof will be provided here.

In a scenario where the probability of detection is constant, the sensor would only have to choose the direction towards where to perform search. Because a particle filter is used, each direction (or sector) $u \in U$ will contain a certain number of particles n_u such that $\sum_{u=1}^{N_U} n_u = N$. Another interpretation of n_u is that it represents the percentage of probability mass that is located in each sector u, given the fact that all the particles have equal weights.

The particle approximations of the two criteria can then be simplified by splitting the sums in two parts: a part where the probability of detection is P_d (i.e. in the chosen sector) and a part where it is zero (i.e. in all the other sectors).

The KL divergence will then be given by:

$$\begin{aligned} KL[\mathbf{q}||\mathbf{p}] &\approx \frac{1}{N}\sum_{j=1}^{N}\frac{1 - P_d(s_k^j, u_k)}{\hat{\mathcal{C}}} \cdot \log\left(\frac{1 - P_d(s_k^i, u_k)}{\hat{\mathcal{C}}}\right) \\ &= \frac{1}{N}\left\{\sum_{j=1}^{n_U}\frac{1 - P_d}{\hat{\mathcal{C}}}\log\left(\frac{1 - P_d}{\hat{\mathcal{C}}}\right) + \sum_{j=1}^{N-n_U}\frac{1}{\hat{\mathcal{C}}}\log\left(\frac{1}{\hat{\mathcal{C}}}\right)\right\} \\ &\dots \\ &= \frac{n_u(1 - P_d) \cdot \log(1 - P_d)}{N - n_u \cdot P_d} + \log(N) - \log(N - n_u \cdot P_d) \end{aligned} \quad (28)$$

and the sector that maximizes Eq. (28) will be chosen.

Accordingly, the second criterion can be simplified as

$$u_k \approx \arg\max_u \left[\frac{1}{N} \sum_{i=1}^{N} P_d(s_k^i, u) \right]$$
$$= \arg\max_u \left[\frac{1}{N} \sum_{i=1}^{n_u} P_d + \frac{1}{N} \sum_{i=1}^{N-n_u} 0 \right]$$
$$= \arg\max_u \left[\frac{n_u}{N} P_d \right] \quad (29)$$

Fig. 14 shows the behavior of the maximum probability of detection based criterion as a function of n_u for various values of the probability of detection. It can be easily noticed that the criterion is a monotonically increasing function of n_u for any value of P_d. This means that the sector that contains the most particles, or equivalently the most probability mass, will be chosen. This can also be inferred by Eq. (29) because N, P_d are constants (known in advance) and therefore they do not affect the sensor management results.

Fig. 2 shows the behavior of the KL based sensor management criterion as a function of n_u for various values of the probability of detection. It is easy to see that it is a monotonically increasing function of n_u for any value of P_d up to a maximum point $maxKL$ that actually depends on P_d. To be more precise, $maxKL$ is assumed for $n_u^{max} \in (N/2, N)$ and the exact value of n_u^{max} depends on P_d.

Therefore, if n_u is lower than n_u^{max} for every $u \in U$ then the two criteria are equivalent because they are both monotonically increasing functions of n_u for any value of P_d. This can be noticed at Fig. 14 and Fig. 2.

On the other hand, if n_u is greater than n_u^{max} then we have to compare the value of $KL(n_u, P_d)$ to the worst case scenario value of $KL(N - n_u, P_d)$ and it actually holds that

$$KL(n_u, P_d) > KL(N - n_u, P_d) \quad , \quad n_u \in (n_u^{max}(P_d), N) \quad (30)$$

Therefore, the two criteria are still equivalent.

The claim that Eq. (30) refers to the worst case scenario can be explained by the fact that $N - n_u \in (0, N/2)$ holds. Therefore, it will also hold that

$$KL(N - n_u, P_d) > KL(n, P_d) \quad (31)$$

for any number of particles n that satisfies $N - n_u > n$ because the KL divergence is a monotonically increasing function for any $n \in (0, N/2)$ and for any P_d.

The conclusion that can be drawn is that both criteria will choose the sector that contains the highest probability mass. Equivalently, if a particle filter approximation is used, they will both choose to search the sector with the largest number of particles.

V. Simulations

A. Constant P_d

The results of the previous section are illustrated by performing 50 Monte Carlo simulations where the sensor has to perform search in 8 sectors with constant $P_d \in (0, 1)$ for $k = 1, \ldots, 160$ sec.

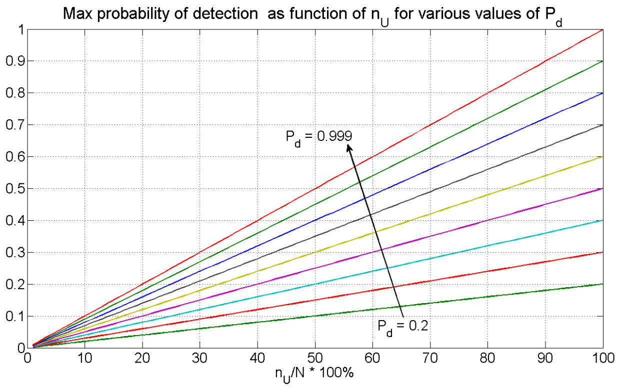

Fig. 1: The behavior of the maximum probability of detection based criterion as a function of n_u for different values of P_d.

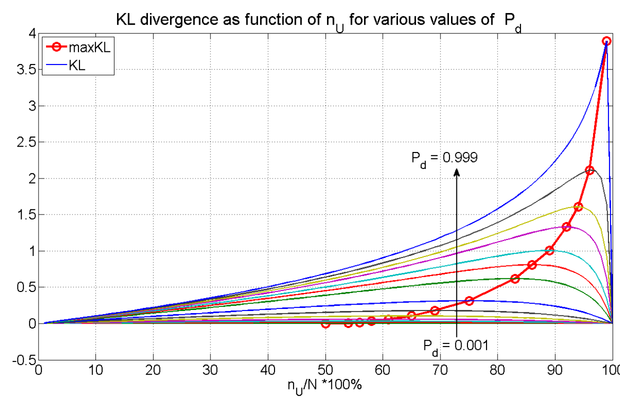

Fig. 2: The behavior of the maximum KL based criterion as a function of n_u for different values of P_d.

An example of such a scenario, where a particle filter approximates the posterior density, is depicted in Fig. 3. The sensor is located at the origin of the axes and it has to choose one of the 8 sectors for performing search. Therefore, the set of sensing actions is equal to set of sectors (8 sectors in this example): $U = \{1, 2, .., 8\}$. Obviously, the probability of detection in the chosen sector is P_d and in all the other sectors is zero. The physical interpretation of this assumption is that we cannot detect the target in sectors that we do not look at.

The density is initialized at $k = 0$ by uniformly distributing the particles in an disk of 100 km radius. The velocities v_x and v_y are chosen such that their vector sum is uniformly distributed in $[0,\ 400]$ m/s towards the direction of the sensor. This initialization process resembles the real life scenario of the moment when the sensor is turned on and there is no information about the target's location, meaning that the target might be anywhere.

For the motion model, we choose $b_x = b_y = 2\ (m/s^2)^2$ as the power spectral densities of the acceleration noise in the $x - y$ direction and $T = 1$ sec as the sampling time.

Furthermore, target birth is modeled at the border of the field of view of the sensor in order to take into account the

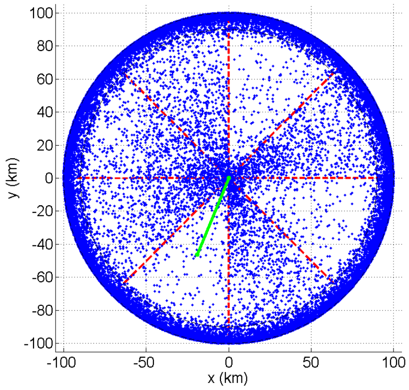

Fig. 3: An example of the density that describes where the undetected target might be. The radar has to search with constant $P_d < 1$ an area of 100 km radius divided in 8 sectors.

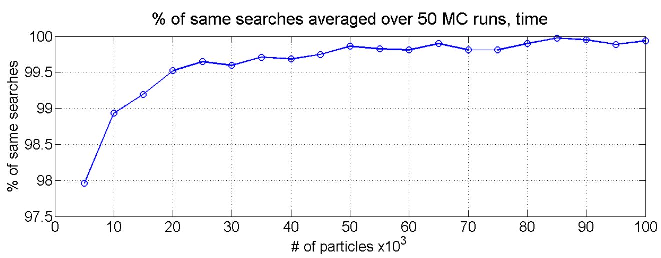

Fig. 4: The percentage of same chosen sensing actions as a function of the number of particles used in the simulations. The results are averaged over 50 MC runs and over the duration of each simulated scenario (160 sec).

fact that the target might have not entered the area yet.

In the simulations, the number of particles is varied such that $N = (5, 10, \ldots, 100) \cdot 10^3$ and we compare the ranking of the sensing actions (in this case sectors) and the percentage of same chosen sensing actions (top ranked sensing actions) of the two criteria. The results are shown in Fig. 4 and Fig. 5.

Fig. 4 shows that as the number of particles increases, the percentage of same chosen sensing actions approaches 100%. Fig. 5 shows that the percentage of differently ranked sensing actions approaches 0% as the number of particles increases. Therefore, the experimental results support the theoretical result that the two sensor management criteria are equivalent.

Another important point is that both criteria produce search patterns that are somehow repetitive and this becomes more obvious as the number of particles used in the simulations increases. Fig. 6 shows an example of a search pattern where this phenomenon can be observed.

B. *Taking into account external information*

We now consider a scenario where the target is expected to be in the 4 northern sectors with 80% probability and in the 4 southern with 20%. All the other parameters in the simulation are the same as the ones used in the previous example.

Fig. 7 demonstrates the adaptiveness of the KL based criterion that focuses on the 4 northern sectors. On the other hand, the simple approach of periodic search wastes time and resources in sectors where the target is not expected to be found with high probability.

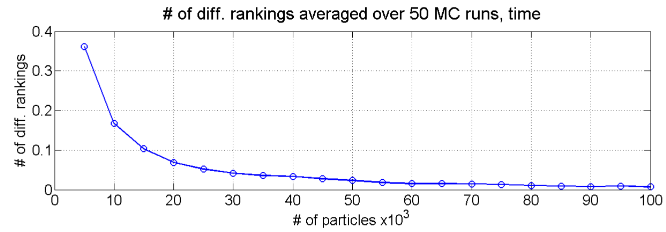

Fig. 5: The percentage of differently ranked sensing actions as a function of the number of particles used in the simulations. The results are averaged over 50 MC runs and over the duration of each simulated scenario (160 sec).

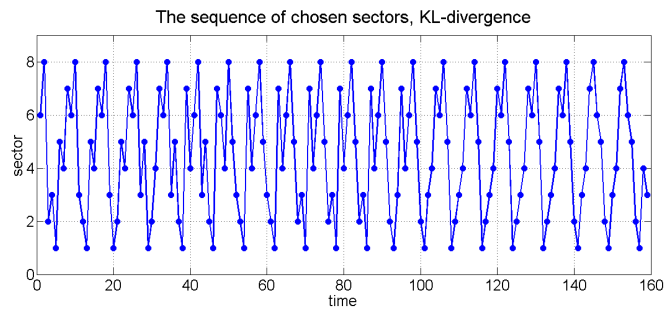

Fig. 6: The search pattern produced by the KL-based criterion for a scenario with constant P_d. It can be noticed that there are several repetitive sub-patterns.

C. *Nonconstant* P_d

In the case of nonconstant P_d we assume that the sensor models the behavior of a multifunction radar. Consequently, P_d depends on the radar cross-section (RCS) of the target and on its distance from the radar.

The rest of the parameters of the scenario are the same, meaning that the radar has to perform search in 8 sectors and that we employ a particle filter with the same dynamical model for the target.

For each particle in the sector to be searched, first the radar equation is used for evaluating the SNR_i:

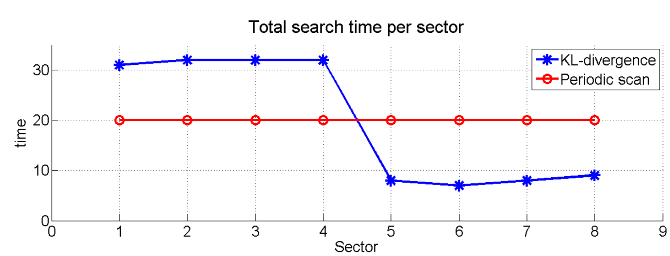

Fig. 7: Search time per sector when the target is expected from the north with 80% probability.

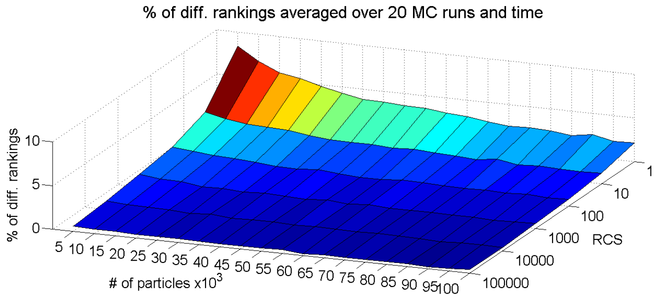

Fig. 8: The percentage of differently ranked sensing actions as a function of the number of particles used for simulation and RCS. The results are averaged over 20 MC runs and over the duration of each simulated scenario (160 sec).

$$
\begin{aligned}
SNR_i\ (dB) = {} & 10\log(P_{peak}) + 10\log(T_{pulse}) + 20\log(\lambda) \\
& + 10\log(RCS_i) + G_{tx} + G_{rx} \\
& - 10\log(k_{Boltzman}) - 10\log(Temp) \\
& - F\cdot L - 10\log[r_i^4(4\pi)^3] \qquad (32)
\end{aligned}
$$

and then the Swerling I case is used for evaluating the corresponding $P_d(i)$ [19]:

$$P_d(i) = P_{fa}^{1/(1+SNR_i)} \qquad (33)$$

where: $r_i = \sqrt{x_i^2 + y_i^2}$, $\lambda = 0.03\ m$, $P_{peak} = 100\ kWatts$, $T_{pulse} = 162\cdot 10^{-6}\ sec$, $G_{tx} = G_{rx} = 35\ dB$, $k_{Boltzman} = 1.37\cdot 10^{-23}$, $Temp = 300\ Kelvin$, $F\cdot L = 1.1\ dB$ losses, probability of false alarms $P_{fa} = 1.4\cdot 10^{-9}$ and $i = 1, 2, \ldots, N$.

Then Eq. (19), (23) and (24) are used for the KL based criterion and Eq. (27) for the maximum probability of detection criterion.

In the experiment, the number of particles is varied such that $N = (5, 10, \ldots, 100)\cdot 10^3$ and the target's RCS is varied such that $RCS = [1\ 10\ 10^2\ 10^3\ 10^4\ 10^5]\ m^2$. We compare the ranking of the sensing actions (again: sectors) and the percentage of same chosen sensing actions (top ranked sectors) of the two criteria. The results are shown in Fig. 8 to 13.

It can be noticed that as the number of particles and the RCS increase, the behavior of the two criteria becomes more similar. The percentage of different rankings approaches 0% and the percentage of same chosen sensing actions approaches 100%. These results indicate that the two criteria are still equivalent in this more involved scenario. Furthermore, the existence of repetitive search sub-patterns was noticed again.

VI. Conclusions

In the previous sections, two fundamentally different sensor management criteria for performing search for one target have been presented and actually shown to be equivalent. This result has two interesting and important implications.

The first implication is the fact that a criterion that is optimal in the information theory context, i.e. maximizing the KL

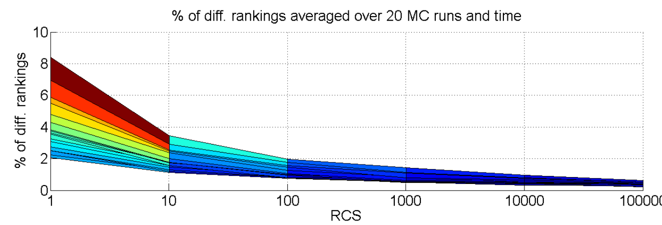

Fig. 9: X-view of Fig. 8.

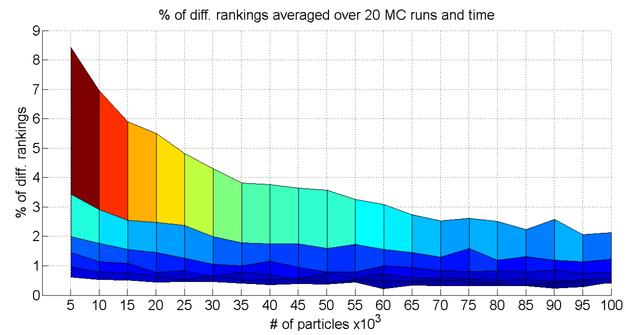

Fig. 10: Y-view of Fig. 8.

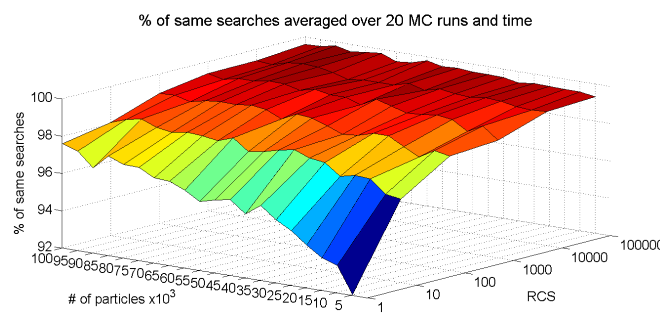

Fig. 11: The percentage of same chosen sensing actions as a function of the number of particles used for simulation and RCS. The results are averaged over 20 MC runs and over the duration of each simulated scenario (160 sec).

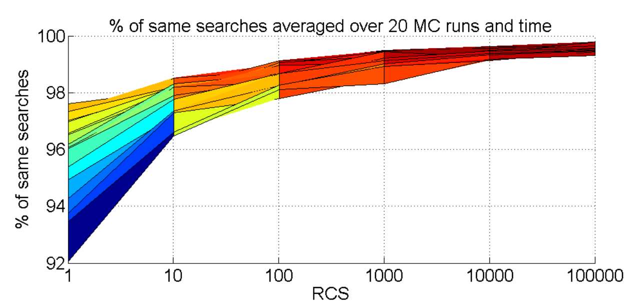

Fig. 12: X-view of Fig. 11.

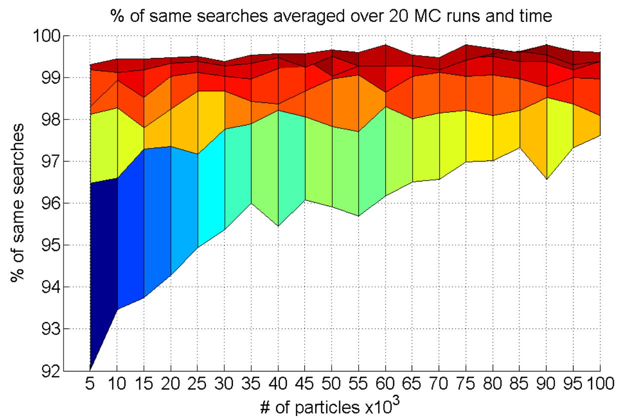

Fig. 13: Y-view of Fig. 11.

divergence between the predictive and the posterior density, is equivalent to a criterion that has straightforward practical and operational meaning, i.e. perform the search action that will yield the maximum expected probability of detecting the target. This means that a criterion that can be easily explained to a person with no background on information theory or filtering is optimal in the information theory context and not just an arbitrarily defined criterion. In other words, it provides a practical interpretation of a criterion that is optimal in the information theory context.

The second implication is that the criterion which is based on the highest probability of detection not only has practical meaning but it is also computationally less expensive to implement, see Eq. (23) and (27). In fact, Eq. (29) means that the implementation of the criterion boils down to just performing a particle count for determining n_u, since N, P_d are constant and known in advance.

Another interesting point is the repetitive sub-patterns that were observed, see Fig. 6. The repetitiveness can be explained by the fact that we assume a uniform distribution of the target density around the border of the area to be searched. The only reason for the search patterns not to be totally repetitive is the randomness induced by the particle filter itself. There is no measurement-induced uncertainty because of the assumption that the measurements indicate that no target has been detected, see subsection III-C.

Some interesting topics that we would like to explore in the future are:

- We would like to compare our approach to other approaches, such as the one presented in [13], in terms of both search results and computational efficiency.
- Another interesting topic is to explore the behavior of the described criteria in multitarget scenaria where external information is also available.
- The presented criteria have certain shortcomings with the most prominent being their difficulty to be tuned in order to meet various operational requirements. Therefore, it appears interesting to explore the usage of sensor management criteria that are based on threat/risk estimation and game theory.

APPENDIX
PROOF OF EQUIVALENCE

The first step is to look at the behavior of the criterion based on the maximum probability of detecting the target. In Fig. 14 and Eq. (29), one can immediately notice that the criterion based on the maximum probability of detecting the target is a monotonically increasing function of n_U for every $P_d \in (0,1)$. As a consequence, the sector with the most particles (or equivalently most probability mass) will be chosen.

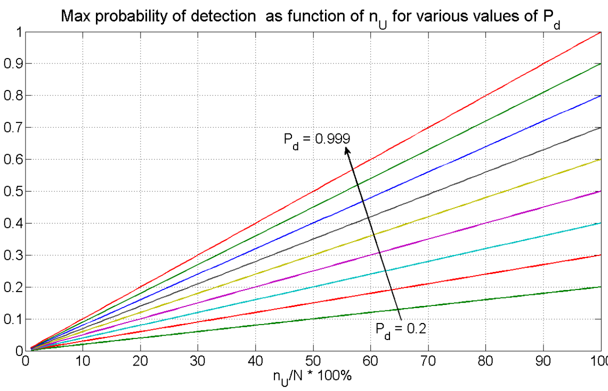

Fig. 14: The behavior of the maximum probability of detection based criterion as a function of n_u for different values of P_d.

Then, for the convenience of the reader, the proof of equivalence is split in two parts.

In the first part it is shown that for $\alpha < \alpha_{cr}(P_d)$, where $\alpha = n_U/N$, the two criteria are equivalent due to the fact that they are both monotonically increasing functions of P_d and α.

In the second and more involved part, it is shown that the two criteria are also equivalent for $\alpha > \alpha_{cr}(P_d)$. In the second part, we will denote by $\alpha_{cr}(P_d)$ a percentage of probability mass (or equivalently, a percentage of the total particles) that is a function of P_d and in any case $1 > \alpha > \alpha_{cr}(P_d) > 1/2$.

The proof that follows is a bit tedious but it boils down to performing monotonicity and sign studies of the involved functions.

A. *Part 1*

Initially, it can be proven that $KL(\alpha, Pd)$ is monotonically non-decreasing for every $P_d \in (0,1)$ by showing that $\partial KL/\partial P_d \geq 0$:

$$\begin{aligned}\frac{\partial KL}{\partial P_d} &= \alpha\left(\frac{1-Pd}{1-\alpha P_d}\right)' \cdot \log(1-P_d) \\ &\quad + \alpha\frac{1-P_d}{1-\alpha P_d}\cdot\frac{-1}{1-Pd} + \frac{\alpha}{1-\alpha P_d} \\ &= \frac{\alpha\left[-(1-\alpha P_d)+\alpha(1-Pd)\right]\cdot\log(1-Pd)}{(1-\alpha P_d)^2} \\ &= \frac{\alpha}{(1-\alpha P_d)^2}\cdot(\alpha-1)\cdot\log(1-Pd) \\ &\geq 0 \end{aligned} \tag{34}$$

because $\alpha/(1-\alpha P_d)^2 \geq 0$, $(\alpha-1) \leq 0$ and $\log(1-Pd) \leq 0$.

Furthermore, it can be shown that there is a series of crucial points of KL in the α-domain. Actually, these happen for $\alpha: 0.5 \rightarrow 1$ as $P_d: 0 \rightarrow 1$. This can be done as follows:

$$\begin{aligned}\frac{\partial KL}{\partial \alpha} &= \log(1-P_d)\cdot\left[\frac{1-P_d}{1-\alpha P_d}+\alpha\cdot\frac{P_d(1-P_d)}{(1-\alpha P_d)^2}\right]\\ &\quad+\frac{P_d}{1-\alpha P_d}\\ &= \frac{(1-P_d)\cdot\log(1-P_d)}{(1-\alpha P_d)^2}+\frac{P_d}{1-\alpha P_d} \end{aligned} \tag{35}$$

If Eq. (35) is set equal to zero and solved for (α) then the crucial points of KL can be obtained:

$$\begin{aligned} 0 &= \frac{\partial KL}{\partial \alpha}\\ \Rightarrow 0 &= (1-P_d)\cdot\log(1-P_d)+P_d\cdot(1-\alpha P_d)\\ \Rightarrow \alpha_{cr} &= \frac{(1-P_d)\cdot\log(1-P_d)+P_d}{P_d^2} \end{aligned} \tag{36}$$

These α_{cr} are increasing as $P_d: 0 \rightarrow 1$ because of Eq. (37), (38) and (39).

$$\begin{aligned}\lim_{Pd\rightarrow 0}\alpha_{cr} &= \lim_{Pd\rightarrow 0}\frac{(1-P_d)\cdot\log(1-P_d)+P_d}{P_d^2}\\ &= \lim_{Pd\rightarrow 0}\frac{-\log(1-P_d)}{2P_d}\\ &= \lim_{Pd\rightarrow 0}\frac{1}{2(1-P_d)}\\ &= 0.5 \end{aligned} \tag{37}$$

$$\begin{aligned}\lim_{Pd\rightarrow 1}\alpha_{cr} &= \lim_{Pd\rightarrow 1}\frac{(1-P_d)\cdot\log(1-P_d)+P_d}{P_d^2}\\ &= \lim_{Pd\rightarrow 1}\frac{(1-P_d)\cdot\log(1-P_d)}{P_d^2}+\lim_{Pd\rightarrow 1}\frac{1}{P_d}\\ &= 0+1\\ &= 1 \end{aligned} \tag{38}$$

$$\begin{aligned}\frac{\partial \alpha}{\partial P_d} &= \frac{[-\log(1-P_d)-1+1]P_d^2}{P_d^4}\\ &\quad-\frac{2P_d[(1-P_d)\cdot\log(1-P_d)+P_d]}{P_d^4}\\ &= \frac{-2P_d-(3-2P_d)\log(1-Pd)}{P_d^3}\\ &> 0 \end{aligned} \tag{39}$$

Ineq. (39) holds because the nominator is positive, which can be shown as follows:

$$\begin{aligned} -2P_d-(3-2P_d)\log(1-Pd) &> 0\\ \log(1-Pd) &< \frac{-2P_d}{(3-2P_d)} \end{aligned} \tag{40}$$

Ineq. (40) holds because:

$$\lim_{Pd\rightarrow 0}\log(1-Pd) = \lim_{Pd\rightarrow 0}\frac{-2P_d}{(3-2P_d)} = 0 \tag{41}$$

and

$$\begin{aligned} [\log(1-Pd)]' &< \left[\frac{-2P_d}{(3-2P_d)}\right]'\\ \frac{-1}{1-Pd} &< \frac{-6}{(3-2P_d)^2}\\ 0 &< 4P_d^2-6P_d+3 \end{aligned} \tag{42}$$

Eq. (42) holds because the determinant of this quadratic polynomial is negative, see Eq. 43, and its second derivative is positive for all P_d.

$$\Delta = \beta^2 - 4\alpha\gamma = 36 - 4\cdot 4\cdot 3 = -12 \tag{43}$$

Fig. (15) shows the crucial points α_{cr} as a function of P_d. Since $KL(a, Pd)$ is zero for $a = 0$ and $a = 1$, $P_d = 0.5$ can be used in order to find a $a_{cr} \simeq 0.6137$. These 2 values can be used to find that $KL(a = 0.6137, Pd = 0.5) = 0.0597 > 0$ and therefore these crucial points are maxima.

At this point it has been shown that if each sector contains less than the crucial number of particles ($a < a_{cr}(P_d)$) then it is straightforward to see that the sector that contains the most particles will be chosen, much like in the case of maximum probability of detection. This happens because for $a \in (0, a_{cr}(P_d))$ the KLD is monotonically increasing for every P_d. Fig. 16 provides a graphical demonstration of the aforementioned claim that has been proven already.

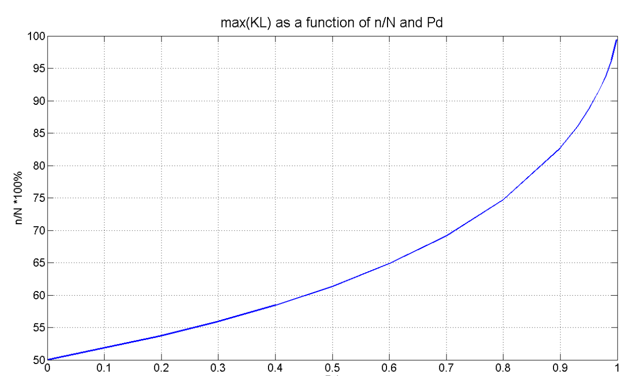

Fig. 15: The combination of $n_U/N = \alpha$ and P_d that lead to $max(KL)$. For $P_d \simeq 0 \Rightarrow max(KL)$ happens for $n \simeq N/2$ and as $P_d: 0 \rightarrow 1$ then KL_{max} happens for $n: N/2 \rightarrow N$.

B. Part 2

We now need to explore what happens if a sector contains more particles than the number that maximizes the value of the KL divergence, meaning $a > a_{cr}(P_d)$. For example if a sector contains 70% of the particles in a scenario with $P_d = 0.5$ then

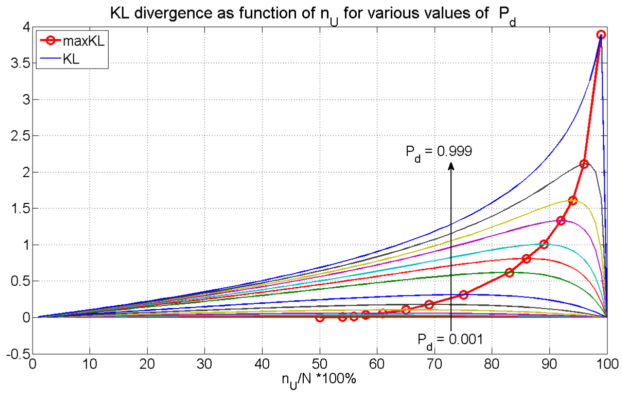

Fig. 16: Graphical proof that $max(KL)$ happens for α_{cr}. The different curves correspond to different values of P_d and the higher the placement of a curve the higher the value of the corresponding P_d.

$$\begin{aligned} &KL(a = 0.7, Pd = 0.5) = 0.0575 \\ \text{but } &KL(a = 0.6137, Pd = 0.5) = 0.0597 \\ \Rightarrow &KL(a = 0.7, Pd = 0.5) < KL(a = 0.6137, Pd = 0.5) \end{aligned} \tag{44}$$

In this case it can be seen that such a comparison, meaning $a = 0.6136$ to $a = 0.7$, does not make sense because there cannot exist at the same time instance two sectors that contain 61.36% and 70% of the particles respectively (the sectors do not overlap).

Therefore, if a sector has $\alpha > \alpha_{cr}$ probability mass, where $\alpha_{cr} > 1/2$, then all the other sectors combined can have up to $1-\alpha < 1/2$ probability mass. Because it has been shown that for any $\alpha < 1/2$ the sector with the most probability mass will be chosen, it follows that the worst case comparison is when a decision between 2 sectors has to be made: a sector with $\alpha > 1/2$ and a sector with $1-\alpha < 1/2$. Therefore, in order to conclude the proof of equivalence of the two criteria, we have to examine if it still holds that the sector with the most particles will be chosen, meaning if Ineq. (45) holds.

$$KL(\alpha, P_d) > KL(1-\alpha, P_d) \quad \text{for} \quad \alpha > 1/2 \tag{45}$$

According to the explanation given above, the sign of Eq. (46) in the interval $\alpha \in [0.5, 1]$ must be studied.

$$\begin{aligned} D(\alpha, P_d) &= KL(\alpha, P_d) - KL(1-\alpha, P_d) \\ &\dots \\ &= \frac{1-P_d}{1-\alpha \cdot P_d} \log(1-P_d) \frac{2\cdot\alpha - 1}{1-(1-\alpha)\cdot P_d} \\ &\quad + \log\left[\frac{1-(1-\alpha)\cdot P_d}{1-\alpha\cdot P_d}\right] \end{aligned} \tag{46}$$

It is straightforward to see that

$$D(\alpha = 0.5, P_d) = D(\alpha = 1, P_d) = D(\alpha, P_d = 0) = 0 \tag{47}$$

Furthermore, using the Symbolics Toolbox of Matlab, it can be shown that $D(\alpha, P_d)$ is a monotonically non-decreasing function for every $P_d \in (0, 1)$ because

$$\begin{aligned} \frac{\partial D(\alpha, P_d)}{\partial P_d} &= -\frac{\alpha \cdot P_d \cdot \log(1-P_d) \cdot (2\cdot\alpha - 1)}{(\alpha \cdot P_d - 1)^2} \dots \\ &\quad \dots \frac{(P_d - 2)\cdot(\alpha - 1)}{(\alpha\cdot P_d - P_d + 1)^2} \\ &> 0 \end{aligned} \tag{48}$$

because P_d, α, $(2\cdot\alpha - 1) > 0$ and $\log(1-P_d)$, $(P_d - 2)$, $(\alpha - 1) < 0$ and the denominator is positive.

Given the pointed out roots of $D(\alpha, P_d)$ and the monotonicity of the P_d component, one has to examine the monotonicity of the α component in order to draw conclusions about the sign of $D(\alpha, P_d)$ in the interval $\alpha \in [0.5, 1]$.

The derivative $\partial D(\alpha, P_d)/\partial\alpha$ (using the Symbolics Toolbox of Matlab) is:

$$\begin{aligned} \frac{\partial D(\alpha, P_d)}{\partial \alpha} &= \frac{P_d}{(\alpha - 1)\cdot P_d + 1} \\ &\quad - \frac{P_d}{\alpha \cdot P_d - 1} + \frac{(P_d - 1)\log(1-P_d)}{\alpha\cdot P_d - 1} \\ &\quad - \frac{(P_d - 1)\log(1-P_d)}{(\alpha - 1)\cdot P_d + 1} \\ &\quad - \frac{\alpha \cdot P_d(P_d - 1)\log(1-P_d)}{(\alpha\cdot P_d - 1)^2} \\ &\quad + \frac{P_d(\alpha - 1)(P_d - 1)\log(1-P_d)}{[(\alpha - 1)\cdot P_d + 1]^2} \end{aligned} \tag{49}$$

and again, if we set Eq. (49) equal to zero we can find a set of crucial points:

$$\begin{aligned} \alpha_{cr} = \pm \Bigg\{ &\Bigg(\frac{\left[8P_d \log(1-P_d)^2 + 4P_d^2 - 4P_d^3 + P_d^4\right.}{Den} \\ &+ \frac{\left.-4\log(1-P_d^2) - 4P_d^2 \log(1-P_d)^2\right]^{1/2}}{Den} \Bigg) \\ &- \Bigg(\frac{0.5 \cdot P_d \left[8P_d \log(1-P_d)^2 + 4P_d^2 - 4P_d^3 + P_d^4\right.}{Den} \\ &- \frac{P_d \left.-4\log(1-P_d^2) - 4P_d^2 \log(1-P_d)^2\right]^{1/2}}{Den} \Bigg) \Bigg\} \\ &+ \frac{1}{2} \end{aligned} \tag{50}$$

where

$$Den = P_d[2P_d - 2\log(1-P_d) + 2P_d \log(1-P_d) - P_d^2]$$

For these 2 sets of crucial points it holds that:

$$\lim_{Pd\to 0} \alpha_{cr}(+) = 0.7887 \quad , \quad \lim_{Pd\to 1} \alpha_{cr}(+) = 1 \tag{51}$$
$$\lim_{Pd\to 0} \alpha_{cr}(-) = 0.2113 \quad , \quad \lim_{Pd\to 1} \alpha_{cr}(-) = 0 \tag{52}$$

and the solution for α_{cr} with positive sign will be chosen because it lies in the interval $\alpha \in [0.5, 1]$ that we consider.

Now it must be shown that these crucial points are increasing $\alpha_{cr} : 0.7887 \to 1$ as $P_d : 0 \to 1$. Therefore, the sign of $\partial\alpha_{cr}/\partial P_d$ is examined:

$$\frac{\partial\alpha_{cr}}{\partial P_d} = -\frac{X}{Y} \tag{53}$$

where the nominator X is given by

$$\begin{aligned} X = {} & 4P_d \log(1 - P_d) - 6P_d^2 \log(1 - P_d) \\ & + 4P_d^3 \log(1 - P_d) - P_d^4 \log(1 - P_d) \\ & + 2[P_d^4 - 4P_d^3 + 4P_d^2] + [8P_d \log(1 - P_d)^2 \\ & - 4P_d^2 \log(1 - P_d)^2 - 4 \log(1 - P_d)^2] \end{aligned} \tag{54}$$

and the denominator Y by

$$\begin{aligned} Y = {} & P_d^2 [2P_d - 2 \log(1 - P_d) + 2P_d \log(1 - P_d) - P_d^2] \cdot \\ & \cdot [P_d^4 - 4P_d^3 + 4P_d^2 + 8P_d \log(1 - P_d)^2 \\ & - 4P_d^2 \log(1 - P_d)^2 - 4 \log(1 - P_d)^2]^{1/2} \end{aligned} \tag{55}$$

1) Nominator sign: The nominator X, see Eq. (54), is *negative* because:

$$\lim_{Pd \to 0} X = 0$$

and its derivative is negative

$$\begin{aligned} \frac{d\ X}{d\ P_d} = {} & 12 \log(1 - P_d) + \frac{1}{P_d - 1} \\ & - P_d \left[8 \log(1 - P_d)^2 + 20 \log(1 - P_d) - 13\right] \\ & - P_d^3 \left[4 \log(1 - P_d) - 7\right] + 8 \log(1 - P_d)^2 + 1 \\ & + P_d^2 \left[12 \log(1 - P_d) - 21\right] \\ < {} & 0\ \forall P_d \in (0, 1) \end{aligned} \tag{56}$$

$d\ X/d\ P_d$ is negative because

$$\lim_{Pd \to 0} \frac{d\ X}{d\ P_d} = 0 \tag{57}$$

and

$$\begin{aligned} \frac{d^2\ X}{d\ P_d^2} = {} & 8 \log(1 - P_d) - 48P_d - \frac{8(P_d^2 - 2P_d + 1)}{(P_d - 1)^2} \\ & - \frac{2(P_d^3 - 4P_d^2 + 6P_d - 4)}{P_d - 1} + 24P_d^2 - 8 \log(1 - P_d)^2 \\ & - \log(1 - P_d) \left[6P_d^2 - 16P_d + 12\right] \\ & - P_d \log(1 - P_d)(6P_d - 8) \\ & + \frac{P_d(P_d^3 - 4P_d^2 + 6P_d - 4)}{(P_d - 1)^2} \\ & - \frac{2P_d(3P_d^2 - 8P_d + 6)}{P_d - 1} \\ & - \frac{16 \log(1 - P_d)(2P_d - 2)}{P_d - 1} + 16 \\ < {} & 0\ \forall P_d \in (0, 1) \end{aligned} \tag{58}$$

$d^2\ X/d\ P_d^2$ is negative because

$$\lim_{Pd \to 0} \frac{d^2\ X}{d\ P_d^2} = 0 \tag{59}$$

and

$$\begin{aligned} \frac{d^3\ X}{d\ P_d^3} = {} & 24 \log(1 - P_d) \\ & - \frac{16 \log(1 - P_d) + P_d^2[16 \log(1 - P_d) + 24]}{(P_d - 1)^3} \\ & - \frac{-P_d[32 \log(1 - P_d) + 48] + 22}{(P_d - 1)^3} \\ & - P_d[24 \log(1 - P_d) - 22] - 22 \\ < {} & 0\ \forall P_d \in (0, 1) \end{aligned} \tag{60}$$

$d^3\ X/d\ P_d^3$ is negative because

$$\lim_{Pd \to 0} \frac{d^3\ X}{d\ P_d^3} = 0 \tag{61}$$

and

$$\begin{aligned} \frac{d^4\ X}{d\ P_d^4} = {} & \frac{16 \log(1 - P_d) + 8}{(P_d - 1)^2} - \frac{6}{(P_d - 1)^4} \\ & - 24 \log(1 - P_d) - 2 \\ < {} & 0\ \forall P_d \in (0, 1) \end{aligned} \tag{62}$$

$d^4\ X/d\ P_d^4$ is negative because

$$\lim_{Pd \to 0} \frac{d^4\ X}{d\ P_d^4} = 0 \tag{63}$$

and

$$\begin{aligned} \frac{d^5\ X}{d\ P_d^5} = {} & \frac{24}{(P_d - 1)^5} - \frac{24}{P_d - 1} - \frac{32 \log(1 - P_d)}{(P_d - 1)^3} \\ < {} & 0\ \forall P_d \in (0, 1) \end{aligned} \tag{64}$$

$d^5\ X/d\ P_d^5$ is negative because

$$\lim_{Pd \to 0} \frac{d^5\ X}{d\ P_d^5} = 0 \tag{65}$$

and

$$\frac{d^6\ X}{d\ P_d^6} = \frac{24}{(P_d-1)^2} - \frac{120}{(P_d-1)^6} + \frac{96\log(1-P_d)-32}{(P_d-1)^4} < 0\ \forall P_d \in (0,1) \tag{66}$$

because for $P_d \in (0,1)$ it holds that

$$\frac{24}{(P_d-1)^2} - \frac{120}{(P_d-1)^6} < 0 \tag{67}$$

and

$$\frac{96\log(1-P_d)-32}{(P_d-1)^4} < 0 \tag{68}$$

2) Denominator sign: The denominator Y, see Eq. (55), is *positive* because $P_d \in (0,1)$ and therefore:

- $P_d^2 > 0$
- $[2P_d - 2\log(1-P_d) + 2P_d\log(1-P_d) - P_d^2] > 0$ since

  $$2P_d - P_d^2 = P_d(2-P_d) > 0$$

  and

  $$2P_d\log(1-P_d) - 2\log(1-P_d) = 2(P_d-1)\log(1-P_d) > 0$$

  because $P_d - 1 < 0$ and $\log(1-P_d) < 0$
- $[P_d^4 - 4P_d^3 + 4P_d^2 + 8P_d\log(1-P_d)^2 - 4P_d^2\log(1-P_d)^2 - 4\log(1-P_d)^2] > 0$

The last point is true because

$$\begin{aligned} Z = {} & P_d^4 - 4P_d^3 + 4P_d^2 + 8P_d\log(1-P_d)^2 \\ & - 4P_d^2\log(1-P_d)^2 - 4\log(1-P_d)^2 \\ = {} & P_d^2(P_d-2)^2 + 4\log(1-P_d)^2[-P_d^2 + 2P_d - 1] \end{aligned} \tag{69}$$

Z is positive because

$$\lim_{Pd \to 0} Z = 0 \tag{70}$$

and its derivative is positive in $P_d \in (0,1)$

$$\begin{aligned} \frac{d\ Z}{d\ P_d} = {} & -4(P_d-1)[P_d(2-P_d) \\ & + 2\log(1-P_d)^2 + 2\log(1-P_d)] \\ > {} & 0 \end{aligned} \tag{71}$$

The derivative $\frac{d\ Z}{d\ P_d}$ is positive because

$$-4(P_d-1) > 0 \tag{72}$$

and

$$V = P_d(2-P_d) + 2\log(1-P_d)^2 + 2\log(1-P_d) > 0 \tag{73}$$

because

$$\lim_{Pd \to 0} V = 0 \tag{74}$$

and

$$\frac{d\ V}{d\ P_d} = 2 - 2P_d + \frac{4\log(1-P_d)+2}{P_d-1} > 0 \tag{75}$$

The derivative $\frac{d\ V}{d\ P_d}$ is positive because

$$\lim_{Pd \to 0} \frac{d\ V}{d\ P_d} = 0 \tag{76}$$

and its derivative is also positive

$$\frac{d^2\ V}{d\ P_d^2} = \frac{2-4\log(1-P_d)}{(P_d-1)^2} - 2 > 0 \tag{77}$$

The second derivative $\frac{d^2\ V}{d\ P_d^2}$ is positive because

$$\lim_{Pd \to 0} \frac{d^2\ V}{d\ P_d^2} = 0 \tag{78}$$

and the third derivative is positive

$$\frac{d^3\ V}{d\ P_d^3} = \frac{8[\log(1-P_d)-1]}{(P_d-1)^3} > 0 \tag{79}$$

because $8[\log(1-P_d)-1] < 0$ and $(P_d-1)^3 < 0$ for every $P_d \in (0,1)$.

3) Sign of $\partial\alpha_{cr}/\partial P_d$: The fact that the nominator is negative and the denominator positive makes Eq. (53) positive, which in turn means that these crucial points are indeed increasing, $\alpha_{cr} : 0.7887 \to 1$ as $P_d : 0 \to 1$.

Given that the crucial points are increasing for $a_{cr} \in (0.5,1)$ and that $D(\alpha_U = 0.5, P_d) = D(\alpha_U = 1, P_d) = D(\alpha_U, P_d = 0) = 0$, one only needs to test the value of D for a specific value of $a_{cr} \in (0.5,1)$ and $P_d \in (0,1)$. If the obtained value of D is positive then the crucial points are maxima and if it is negative then they are minima. We test for $a_{cr} = 0.8$ and $P_d = 0.5$ and it holds that:

$$D(\alpha_U = 0.8, P_d = 0.5) = 0.02038 > 0 \tag{80}$$

therefore the crucial points are maxima and $D > 0$ for $a_{cr} \in (0.5,1)$ and $P_d \in (0,1)$.

C. Conclusion

Combining the 2 parts of the proof means that the two compared criteria, i.e. choosing the sensing action that maximizes the Kullback-Leibler divergence or the action that yields the maximum probability of detecting a target, produce the same sensor management results when performing search with constant probability of detection.

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References

[1] J. M. Danskin, "A helicopter versus submarine search game," *Operations Research*, vol. 16, no. 3, pp. 509–517, 1968.

[2] B. O. Koopman, "The Theory of Search. III. The Optimum Distribution of Searching Effort," *Operations Research*, vol. 5, no. 5, pp. 613–626, Oct. 1957. [Online]. Available: http://www.jstor.org/stable/167462

[3] T. M. Kratzke and J. R. Frost, "Search and rescue optimal planning system," in *Proceedings of the 13th International Conference on Information Fusion*, vol. 1, 2010, pp. 1–9.

[4] L. Stone, C. Keller, T. Kratzke, and J. Strumpfer, "Search analysis for the underwater wreckage of Air France Flight 447," in *Proceedings of the 14th International Conference on Information Fusion*, vol. 1, 2011, pp. 1–8.

[5] I. Suzuki and M. Yamashita, "Searching for a mobile intruder in a polygonal region," *SIAM Journal on Computing*, vol. 21, no. 5, pp. 863–888, 1992.

[6] B. P. Gerkey, S. Thrun, and G. Gordon, "Visibility-based pursuit-evasion with limited field of view," in *International Journal of Robotics Research*, 2004, pp. 20–27.

[7] F. Bolderheij and P. Van Genderen, "Mission driven sensor management," in *Proceedings of the 7th International Conference on Information Fusion*, 2004.

[8] W. Koch, "On adaptive parameter control for phased-array tracking," in *Proceedings of Signal and Data Processing of Small Targets*, 1999.

[9] ——, "On exploiting 'negative' sensor evidence for target tracking and sensor data fusion," *Information Fusion*, vol. 8, no. 1, pp. 28–39, 2007.

[10] K. White, J. Williams, and P. Hoffensetz, "Radar sensor management for detection and tracking," in *Proceedings of the 11th International Conference on Information Fusion*, 2008, pp. 1–8.

[11] Y. Boers, H. Driessen, and L. Schipper, "Particle filter based sensor selection in binary sensor networks," in *Proceedings of the 11th International Conference on Information Fusion*, 2008, pp. 1–7.

[12] D. Matthiesen, "Efficient beam scanning, energy allocation, and time allocation for search and detection," in *IEEE International Symposium on Phased Array Systems and Technology (ARRAY)*, oct. 2010, pp. 361 –368.

[13] A. Charlish, K. Woodbridge, and H. Griffiths, "Agent based multi-function radar surveillance control," in *Proceedings of the IEEE Radar Conference (RADAR)*, vol. 1, 2011, pp. 824 – 829.

[14] B. Ristic, S. Arulampalam, and N. Gordon, *Beyond the Kalman filter*. Artech House, 2004.

[15] X.-L. Hu, T. Schon, and L. Ljung, "A basic convergence result for particle filtering," *IEEE Transactions on Signal Processing*, vol. 56, no. 4, pp. 1337 –1348, april 2008.

[16] E. Aoki, A. Bagchi, P. Mandal, and Y. Boers, "A theoretical look at information-driven sensor management criteria," in *Proceedings of the 14th International Conference on Information Fusion*, vol. 1, 2011, pp. 1180–1187.

[17] A. Doucet, B. Vo, C. Andrieu, and M. Davy, "Particle filtering for multi-target tracking and sensor management," in *Proceedings of the 5th International Conference on Information Fusion*, 2002, pp. 474–481.

[18] Y. Boers, H. Driessen, A. Bagchi, and P. Mandal, "Particle filter based entropy," in *Proceedings of the 13th International Conference on Information Fusion*, 2010.

[19] M. I. Skolnik, *Introduction to Radar Systems*, 3rd ed. McGraw-Hill Science/Engineering/Math, 2002.