Multi-Objective Road Pricing: A Game Theoretic and Multi-Stakeholder Approach

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Abstract

Costs associated with traffic externalities such as congestion, air pollution, noise, safety, etcetera are becoming “unbearable”. The Braess paradox shows that combating congestion by adding infrastructure may not improve traffic conditions, and geographical and/or financial constraints may not allow infrastructure expansion. Road pricing presents an alternative to combat the mentioned externalities. The traditional way of road pricing, namely; congestion charging, may create negative benefits for the society and stakeholders, thus, defeating its main purpose (increasing transportation efficiency and social welfare). We study a road pricing that encompasses all the mentioned externalities. A meanwhile standard approach to deal with conflicting objectives (externalities) are models from Multi-objective Optimization. This approach assumes that there is one leader stakeholder/decision-maker. But then, if more than one stakeholder participates in the road pricing, the concept of Nash equilibrium (NE) from economics may constitute an alternative model. Using game theoretic approach, we study and extend the single authority road pricing scheme (Stackelberg game) to a pricing scheme with multiple authorities/regions (with likely contradicting objectives). Our model includes users interests in the upper level - giving a promising model that deals with user acceptability of road pricing. We investigate the existence of NE among actors and prove that no pure NE exists in general. Then again, NE may exist under special conditions. Since NE may not exist, and since competition may deteriorate the social welfare, we further design a mechanism that simultaneously induces a pure NE and cooperative behaviour among actors, thus, yielding optimal tolls for the system.

1 INTRODUCTION

Over the past years, vehicle ownership has increased tremendously. It has been realized that the social cost of owning and driving a vehicle does not only include the purchase, fuel, and maintenance fees, but also the cost of man hour loss to congestion and road maintenance, costs of health issues resulting from accidents, exposure to poisonous compounds from exhaust pipes, and high noise level from vehicles. So, to optimize the traffic flow requires a model that optimizes more than one objective which may be in conflict with each other. The model should also maximize the user benefit. Optimization of more than one traffic externality is not a novel idea. Road pricing that simultaneously treat time losses, increased fuel consumption, and emission is discussed in [1, 2]. Traffic congestion, air pollution and accident externalities are considered in [3]. Single- and bi-criteria Pareto optimization that deal with users with different values of time and two objectives (time and money) were studied in [4, 5, 6]. Road damage externality is incorporated in the road pricing models of [7].

All the models mentioned above are based on the idea of multi-objective optimization where one leader decides which point on the Pareto-front is chosen. They all have one shortcoming; they do not address the issues arising when different stakeholders/autonomous cities with possibly conflicting objectives toll the road. There is need for such models since autonomy of states/cities or regions are increasing becoming popular in the area of infrastructure or road management. In literature, there are few works dealing on these shortcomings, competition among stakeholders (we use the terms stakeholders, leaders and actors interchangeably) with privately owned network with intention of maximizing their toll revenue is studied in [8, 9]. They formulated their problem as equilibrium problem with equilibrium constraints (EPEC). Both toll and capacity competition among private asymmetric roads with congestion in a network with parallel links is studied in [10]. In their paper, [10] analyzed the allocative efficiency of private toll roads vis a vis free access and public toll road pricing on a network with two parallel routes joining a common origin and destination. In one of their study regimes, they considered a mixed duopoly with a private road competing with a public toll road. On the other hand, tax competition on a parallel road network when different governments have tolling authority on the different links of the network is studied in [11].

The studies mentioned in the foregoing literature assume that network or road segments are privately owned or managed by private stakeholders. They do not take into account that private stakeholders (with likely contradicting objectives) that do not own networks influence the implementation of road pricing (or nature of tolls) during policy making. Again, users acceptability of road pricing was not discussed; users were modeled to have no say on the imposed tolls. Campaigns on the implementation of road pricing have failed in many cities like Edinburgh (in 2002), Trondheim (in 2005), New York (in 2008), Hong Kong (in 1986), cities in the Netherlands, just to mention a few, due to lack of support. This lack of support is due to the fact that the debate on the implementation comprises of stakeholders of conflicting interests and users are most times never considered on the same level as these stakeholders. In this paper, we address these issues and formulate a general model that allows each stakeholder (including users) partake in toll setting. In our models, the tolls are used to maximize stakeholder’s or system’s social welfare and not as a way of generating revenue. We assume that the tolls are returned back into the transportation system...
so as not to increase the societal cost. Users are modeled on the same level as the stakeholders with one stakeholder representing users’ interest.

The rest of the paper is organized as follows: section 2 gives the basic traffic model for our road pricing problem and extend the usual single leader single-objective road pricing to single leader multi-objective road pricing. Section 3 then extends the single leader to multi-leader multi-objective road pricing using game theoretical approach; introducing the concept of Nash equilibrium for the road pricing game. In section 4, we introduce the optimal Nash inducing mechanism which ensures that Nash equilibrium exists, and that it coincides with system optimum. We demonstrate the models using numerical examples in section 5. Section 6 concludes the paper.

2 BASIC TRAFFIC MODEL FOR ROAD PRICING

2.1 Notations

Let $G=(N,A)$ be a network, with $N$ the set of all nodes in the network and $A$ the set of (directed) arcs or links in $G$. We use the following notation:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>set of all arcs (links) in $G$</td>
</tr>
<tr>
<td>$R$</td>
<td>set of all paths in $G$</td>
</tr>
<tr>
<td>$W$</td>
<td>set of all OD pairs in $G$</td>
</tr>
<tr>
<td>$f$</td>
<td>path flow vector in $G$</td>
</tr>
<tr>
<td>$v$</td>
<td>vector of link flows in $G$</td>
</tr>
<tr>
<td>$d$</td>
<td>travel demand vector in $G$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>arc-path incident matrix in $G$</td>
</tr>
<tr>
<td>$D(\lambda)$</td>
<td>vector of demand functions in $G$</td>
</tr>
<tr>
<td>$B(d)$</td>
<td>inverse demand (or benefit) function</td>
</tr>
<tr>
<td>$\lambda_{lc}$</td>
<td>least cost to transverse the $w^{th}$ OD pair</td>
</tr>
<tr>
<td>$e(v)$</td>
<td>vector of link emission functions in $G$</td>
</tr>
<tr>
<td>$s(v)$</td>
<td>vector of safety functions in $G$</td>
</tr>
<tr>
<td>$n(v)$</td>
<td>vector of link noise functions in $G$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>vector of monetary value of emission per gramme depending on say urbanization</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>monetary equivalent of $1dB(A)$ defined for a certain noise level in $G$</td>
</tr>
<tr>
<td>$K$</td>
<td>set of all actors in the road pricing game</td>
</tr>
<tr>
<td>$C_k(v)$</td>
<td>total network cost function for the $k^{th}$ objective in $G$, with $C_k(v) = \sum_{a \in A} C_k^a(v_a)$</td>
</tr>
<tr>
<td>$C$</td>
<td>vector of network cost functions in $G$</td>
</tr>
<tr>
<td>$Z(v)$</td>
<td>total network cost in $G$, i.e. $Z(v) = \sum_{k \in K} C_k(v)$</td>
</tr>
</tbody>
</table>

2.2 Single Leader Problem Formulation

2.2.1 Stakeholder’s Problem

We summarize the “tolling problem” for elastic demand where each stakeholder $k$ would like to solve as if he were the unique leader. We assume that each stakeholder controls a unique objective, and he wishes to maximize the user benefit and to minimize his own costs $C_k(v)$ under user flow and environmental feasibility conditions. We have also assumed a unimodal model, a multimodal model is straightforward by adding a superscript on each flow related (dependent) entity, parameter and/or variable to indicate the user class. By Beckmann’s formulation [12] the user benefit (UB) is given by

$$UB = \sum_{w \in W} d_w \int_{0}^{\infty} B_w(\zeta) d\zeta$$

where $B_w(d_w)$ is the inverse demand or benefit function for the OD pair $w \in W$. Observe that $UB = 0$ when the demand is fixed.

Assuming that UB is split equally among all stakeholders, the problem of stakeholder $k$ can then be
stated as follows:

\[
\begin{align*}
    SP_k : \quad \min_{v,d} Z_k := C_k(v) &= \frac{1}{|K|} \sum_{w \in W} d_w \int_0^{|w|} B_w(\xi) d\xi \\
    \quad \text{s.t.} & \quad \begin{cases} 
        \Gamma f = d \lambda \\
        \delta f \geq 0 \\
        g(v) \leq 0 \\
        \phi \leq 0 \\
        \psi \leq 0 \\
        \xi \leq 0 \\
        \rho \leq 0 \\
    \end{cases} \\
    \quad \text{(FeC-ED)} \tag{1}
\end{align*}
\]

Here, \(|K|\) denotes the number of stakeholders. The first set of constraints is the flow feasibility conditions for elastic demand (FeC-ED); the first constraint states that the flow on a link is equal to the sum of all path flows that passes through this link, the second equation states that the sum of flows on all paths originating from origin node \(\rho\) and ending at destination node \(q\) for an OD pair \(pq\) equals the demand for this OD pair, the third and fourth inequalities simply state that the path flows (and thus the link flows) and \(g(v)\) functions are separable and strictly monotonic, and that the side constraints in the KKT conditions (see Eqn 1). The last constraint \(g(v) \leq 0\) (where \(g(v) \in \mathbb{R}^{|A| \times |K|}\)) contains possible side constraints on the link flow vector \(v\). These side constraints (which we assume to be convex or linear in \(v\)) may be standardization constraints such as:

- The total emission on certain links should not exceed the stipulated emission standard.
- The total noise level on certain links should not exceed the standard allowed dB(A) level.
- The number of cars on certain roads should not exceed certain numbers so as to preserve the pavements and check accidents, etcetera.

**Assumption 1:** Throughout (and for easiness) we assume that the link cost (travel time) functions are separable, that all functions \(C_k^q(v_a)\) in the objective \(C_k(v)\) are strictly convex in \(v_a\), that the inverse demand functions are separable and strictly monotonic, and that the side constraints \(g(v) \leq 0\) are linear.

### 2.3 Multi-objective Model (MO)

In a standard MO model that considers all stakeholders, one has to solve \([13, 14]\) a program such as:

\[
\min_{v,d} Z = (SP_1, SP_2, SP_3, SP_4, \ldots) \quad \text{s.t.} \quad \text{FeC-ED} \tag{2}
\]

Where the indices, \((t, e, n, s, i, \ldots)\) refer to different objectives (see for example the table in section 2.1). More precisely, one has to find a point on the Pareto front of this program. In what follows we will consider the Pareto point given as the minimizer of the (special) MO program (system monetary costs \(Z = \sum_{k \in K} SP_k\)):

\[
\begin{align*}
    MO : \min_{v,d} Z := & \sum_{k \in K} C_k(v) - \sum_{w \in W} d_w \int_0^{|w|} B_w(\xi) d\xi \\
    \text{s.t.} & \quad \text{FeC-ED} \tag{3}
\end{align*}
\]

Note that by choosing different weight factors for the objectives in the MO, we can model preferences for some externalities.

#### 2.3.1 (Road) User Problem - UP

Without loss of generality, we assume that the only determinant of user’s route choice behaviour is the travel costs and benefits of a trip. Under Assumption 1, the well-known Beckmann’s formulation of Wardrop’s user equilibrium (UE) \([12]\) describes the users’ behaviour mathematically by the convex program:

\[
\begin{align*}
    UP : \min_{v,d} & \sum_{a \in A} v_w \int_0^{v_w} \beta_{wA} u du - \sum_{w \in W} d_w \int_0^{|w|} B_w(\xi) d\xi \\
    \text{s.t.} & \quad \text{FeC-ED}
\end{align*}
\]

### 2.4 First and Second-best Pricing

To solve the toll pricing problem in presence of one leader, first and second best pricing techniques are mostly used. The first-best pricing idea is based on a comparison between the KKT-conditions for MO and the KKT-conditions for UP. In general the first best prices are not unique. We summarize the result in the following corollary (see \([15]\) for proof, and \([16]\) for a similar result).
Corollary 1
Suppose \( \langle \bar{v}, \bar{d} \rangle \) is a solution for the MO, then any social toll vector \( \theta \) (with toll \( \theta_a \) on link \( a \)) satisfying the following set of linear conditions is a toll such that \( \langle \bar{v}, \bar{d} \rangle \) is also the elastic user equilibrium with respect to costs \( \beta_t(v) + \theta \):

\[
\begin{align*}
\sum_{a \in A} (\beta_t(v_a) + \theta_a) d_{ar} & \geq B(d_{ar}) \quad \forall r \in R, \forall w \in W \\
\sum_{a \in A} (\beta_t(v_a) + \theta_a) \bar{v}_a & = \sum_{w \in W} B(d_w) d_w
\end{align*}
\]

for short:\[
\begin{align*}
\Lambda^T (\beta_t(v) + \theta) & \geq \Gamma^T B(d) \\
(\beta_t(v) + \theta)^T \bar{v} & = (\bar{d})^T \bar{d}
\end{align*}
\]

We will refer to Equation (4) as equilibrium constraint for elastic demand (EqC_ED). For fixed demand, the matrix form of Equation (4) becomes:

\[
\Lambda^T (\beta_t(v) + \theta) \geq \Gamma^T \lambda
\]

\[
(\beta_t(v) + \theta)^T \bar{v} = (\bar{d})^T \lambda
\]

where \( \lambda \) is a free vector (of multipliers, see Eqn 1) with components \( \lambda_w \) representing the minimum route travel cost for a given OD pair. One of the possible tolls is given by the “first pricing” toll (see [15, 16] for proofs):

\[
\theta_{sc} = \sum_{k \in K} |K| \nabla C_k(\bar{v}) - \beta_t(\bar{v}) + |K| \nabla g(\bar{v}) \xi
\]

If there are extra conditions on the toll vector \( \theta \) (e.g., some links \( a \in Y \) are non-tollable (\( \theta_a = 0 \)) there might be no feasible first-best pricing toll. In this case one has to find a second-best pricing vector, and instead of solving a standard program MO one has to solve the following bi-level program also called a mathematical program with equilibrium constraints (MPEC):

\[
\min_{d, v, \theta} Z = \sum_k C_k(v) - \sum_{w \in W} \int_0^{d_w} B_w(\xi) d\xi
\]

subject to:

\[
\begin{align*}
\Lambda^T (\beta_t(v) + \theta) & \geq \Gamma^T B(d) \\
(\beta_t(v) + \theta)^T v & = (\bar{d})^T d \\
\theta_a & = 0 \forall a \in Y \\
FeC_ED
\end{align*}
\]

3 MULTI-LEADER MODEL IN ROAD PRICING

In the foregoing models, we discussed a one leader road pricing using the MO program. Such models have their shortcomings; when one decision maker (dm), (e.g. the government) controls the traffic flow of a transportation system through road pricing, then it is likely that some other stakeholders affected by activities of transportation may not be happy with the decisions made by this dm. This is because when the dm models the MO road pricing problem, all traffic externalities are simultaneously considered with or without preference for any externality (see MO Eqn 3). When preference is given, say, to congestion, then the effect of the preferred externality subdues the effect of other externalities, and this may translate to huge costs for some stakeholders. For example, lower travel time (say high speeds) may translate to more accidents (costs for insurance companies). Even without preference to any externality, it is intuitive that stakeholders still will prefer to partake in toll setting to safeguard their interests. The main problem of a classical approach from multi-objective optimization is the following: supposing that each stakeholder can influence the toll setting, why should a (independent) player accept a situation which he can improve by changing the tolls?

In such a situation the classical concept of Nash equilibrium in game theory gives an appropriate alternative model. Such models are accepted in economics in situations where independent players may influence the market with their strategies in order to optimize their specific objective.

The question we like to address from game theoretical/economic point of view is; What happens when each stakeholder optimizes his objective by tolling the network, given that other stakeholders are doing the same? Formally, we introduce the mathematical and economic theory behind.

3.1 Mathematical and Economic Theory

The MPEC (Eqn (7)) described in the previous section is a Stackelberg game where a leader (dm) moves first followed by sequential move of other players (road users). If we assume that various stakeholders are allowed to set toll (or at least influence the tolls) on the network, then, users are influenced not only by just one leader as in Stackelberg game, but by more than one decision maker. In a multi-leader-multi-follower game/problem, the leaders take decisions (search for toll vectors \( \theta^k, k \in K \), that optimize their respective
objects) at the upper level which influence the followers (users) at the lower level. The followers then react accordingly (user/Wardrop equilibrium), which in turn may cause the leaders to update their individual decisions leading to lower level players reactions again. These updates continue until a stable situation is reached. A stable state is reached if no stakeholder can improve his objective by unilaterally changing his toll. Note however, that given the stable state decision tolls of leaders, the lower level stable situation is given by the (unique) Wardrop’s equilibrium. So the bi-level game can be seen as a single (upper) level game with additional equilibrium conditions (for the lower level).

In the above scenario, each actor continuously solves an MPEC which is influenced by other actors’ MPECs, and this translates to an equilibrium problem subject to equilibrium constraints (EPEC). Since a stable state upper level tolls will lead to a (unique) Wardrop’s equilibrium in the lower level, our aim therefore is to find a Nash toll vector for the leaders (see figure 1).

After settling on a Nash toll vector, users represented in the upper level search for an alternative but lower toll vector using Equation (4).

Remark: The theory described above does not necessarily mean that stakeholders have different toll collecting machines on the links. Our model describes the Nash toll vector that can be agreed upon during policy making or debate.

![FIGURE 1 Multi-Leader-Multi-Follower Nash/Cooperative Game Model](image)

### 3.2 Mathematical Models for the Bilevel Nash Equilibrium Game (EPEC)

We now mathematically introduce the toll pricing game and the concept of Nash equilibrium (NE) [17, 18] as outlined in subsection 3.1.

Assume that Assumption 1 holds. This in particular ensures that (for given costs) the Wardrop equilibrium (WE) $(v, d)$ is unique. Let $\theta^k$ be the link toll vector of player $k \in K$. We use $\theta^{-k}$ to denote all toll vectors in $K \setminus k$. In the Nash game, for given $\theta^{-k}$, the $k^{th}$ stakeholder tries to find a solution toll $\hat{\theta}^k$ for the following problem:

$$\psi_k(\hat{\theta}^k, \hat{\theta}^{-k}) = \min_{\theta^k} \psi_k(\theta^k, \hat{\theta}^{-k})$$

where for given $\theta^k$ (and $\hat{\theta}^{-k}$)

$$\psi_k(\theta^k, \hat{\theta}^{-k}) := \min Z_k = C_k(v^k) - \frac{1}{|K|} \sum_{w \in W} \int_0^{d_w} B_w(\zeta) d\zeta \quad s.t$$

$$\Lambda^T \left( \beta_l(v^k) + \theta^k + \sum_{j \in K \setminus k} \hat{\theta}_j^k \right) \geq \Gamma^T B(d^k)$$

and

$$\beta_l(v^k) + \theta^k + \sum_{j \in K \setminus k} \hat{\theta}_j^k = B(d^k)^T (d^k)$$

(8)

The concept of a Nash equilibrium is to look for a situation where for fixed strategies $\hat{\theta}^{-k}$ of the opponent players, the best that player $k$ can do is to chose his own toll to be $\hat{\theta}^k$. A NE is thus a whole set of toll vectors $\hat{\theta} = (\hat{\theta}^k, k \in K)$ such that for each player $k$ the following holds:

$$\psi_k(\theta^k, \hat{\theta}^{-k}) \leq \psi_k(\hat{\theta}^k, \hat{\theta}^{-k}) \quad \text{for all feasible toll } \theta^k \text{ and } \forall k \in K$$

(9)

See that in the optimization problem above, each leader $k$ can only change his own link toll vector $\theta^k$. The strategies $\hat{\theta}_j^k, j \neq k$ of the other leaders are fixed in $k$’s problem. The left hand constraints are the equilibrium constraints and the right ones are the feasibility conditions.
3.3 Existence of Nash Equilibrium

In this subsection we analyse the existence of Nash equilibrium in our tolling game. We show below that this simple standard Nash equilibrium concept (Eqns (8) & (9)) is not directly applicable to the tolling problem. The main reason lies in the special structure of the problems \( \Psi_k(\ell^k, \ell^{-k}) \) in Equation (8) leading to the following fact:

**Fact:** Due to Assumption 1, for given vectors \( \ell^k, k \in K \) the corresponding solution \( (\ell^*, \ell^{**}) \) of the system (8) (i.e., the elastic demand user equilibrium with respect to the costs \( \beta t(l) + \sum_{k \in K} \ell^k \)) is uniquely given. Therefore it holds:

If \( \ell^* \) is a Nash equilibrium, then all corresponding solution vectors \( (\ell^k, d^k) = (\ell^*, d^*) \), \( k \in K \) of \( \Psi_k \) are identical. (10)

**Proof:** Given that \( \ell^* \) solves system (8) for all actors \( k \in K \), then it means that at Nash equilibrium among the actors, the link toll vector \( \ell \) is given by \( \ell = \sum_{k \in K} \ell_k \), where \( \ell_a = \sum_{k \in K} \ell^k_a, \forall a \in A \). Due to Assumption 1, this toll vector \( \ell \) yields a unique flow pattern \( (\ell^*, d^*) \). Of course the users do not differentiate the tolls (per actor \( k \)), what they experience is the total toll vector \( \ell \), and as such, the vector \( \ell \) (together with the travel time costs) determines the unique user/Wardrop equilibrium flow \( (\ell^*, d^*) \) for the system.

3.3.1 Unrestricted Toll Values

From the relation (10) we can directly deduce the following results.

**Corollary 2**

(a) Suppose the leaders can toll all links with no restrictions (no constraint \( \ell^k \geq 0 \) in (Eqn 8)), then, for the tolling game with elastic demand, there does not exist a Nash equilibrium in general. Moreover, in this game the players do not have any incentive to cooperate.

(b) When the demand is fixed, even under the extra conditions \( \ell^k \geq 0 \) in Equation (8), there does not exist a Nash equilibrium in general.

**Proof:** We will even show that in (the general) case where not all players have the same solution \( (\ell^k, d^k) \) in their own program \( SP_k \) (see Eqn 1) there will never be a Nash equilibrium of the form in Equation (9).

(a) Assume \( \ell \) is a Nash equilibrium with \( (\ell^*, d^k) \) the solution of player \( k \). Recall that (by Eqn 10) all user flows \( (\ell^*, d^*) \) are the same at Nash. By assumption, at least one of the players, say player \( l \), has a different ideal (or optimal) link flow \( (\ell^*, d^*) \) in \( SP_k \) (since players are assumed to have conflicting objectives) and by our discussion in Section 2, player \( l \) can achieve this flow in \( \Psi_l(\ell^*, \ell^{-l}) \) by choosing e.g., the first best pricing toll

\[
\ell^l = |K| \nabla C_l(\ell) - \beta t(\ell) - \sum_{k \in K \setminus l} \ell^k
\]

where \( |K| \) is the number of players. Note that this toll \( \ell^l \) may be negative. Since at any stage of the game, any player \( k \) can always achieve his ideal flow in \( SP_k \), it is clear that no equilibrium can be reached and that players do not have any reason for cooperate if they can always achieve \( SP_k \) on their own.

(b) The same clearly holds in the case of fixed demand. However, in this case we can always achieve a first best pricing toll in Equation (5) satisfying \( \ell^l \geq 0 \). To see this, note that for fixed demand, any leader \( l \in K \) has the following valid toll vectors as part of a whole polyhedron (see proof below) that achieve the ideal flow vector for leader \( l \) [19]

\[
\ell^l = [\alpha (\nabla C_l(\ell)) - \beta t(\ell)] - \sum_{k \in K \setminus l} \ell^k; \quad \text{where} \quad \alpha > 0
\]

By making \( \alpha \) large enough we can assure \( \ell^l \geq 0 \).

**Proof:** Suppose \( \ell \) is an ideal flow vector that solves (1) (omitting the UB - fixed demand) for player \( l \), now let \( \ell^l \) be the corresponding toll vector satisfying (5), this means that \( \ell \) is solution of the LP

\[
\min_v (\beta t(\ell) + \ell^l)^T v \quad s.t. \quad v \in V
\]

where \( \beta t(v) \) is a vector of link travel time functions. Obviously \( \ell \) also solves the following LP

\[
\min_v \alpha (\beta t(\ell) + \ell^l)^T v \quad s.t. \quad v \in V \quad \text{where} \quad \alpha > 0
\]
but,

\[
\alpha (\beta_t(\bar{v}) + \theta^i)^T v = \left( \left( \beta_t(\bar{v}) + \theta^i \right) + (\alpha - 1) \left( \beta_t(\bar{v}) + \theta^i \right) \right)^T v
\]

\[
= \left( \beta_t(\bar{v}) + \left[ \theta^i + (\alpha - 1) \left( \beta_t(\bar{v}) + \theta^i \right) \right] \right)^T v
\]

this means that with \( \theta^i \), any vector

\[
\tilde{\theta}^i = \theta^i + (\alpha - 1) \left( \beta_t(\bar{v}) + \theta^i \right) = \alpha \left( \beta_t(\bar{v}) + \theta^i \right) - \beta_t(\bar{v})
\]

is a valid toll vector as well. Recall that for fixed demand and for one objective, the marginal social cost (MSC) toll or the so called first best toll given by (see Eqn (6) - elastic demand equivalent):

\[
\theta^i = \nabla G_i(\bar{v}) - \beta_t(\bar{v})
\]

is one toll vector that achieves the ideal flow vector \( \bar{v} \), therefore

\[
\tilde{\theta}^i = \alpha (\nabla G_i(\bar{v})) - \beta_t(\bar{v}) - \sum_{k \in K \setminus \ell} \tilde{\theta}^k; \text{ where } \alpha > 0 \quad \Box
\]

We emphasize that extra restrictions on the tolls \( \theta^k \) may play in favor of the existence of a Nash equilibrium.

Generally, what can we say on the existence of NE? A well-known theorem in game theory [20] states that the game has a Nash equilibrium if the following conditions are met:

- The strategy sets for each player are compact and convex, and each player’s utility function \( \Psi_k(\theta^k, \tilde{\theta}^{-k}) \) is continuous and quasi-convex in his strategy \( \theta^k \).

However, in general we cannot expect such a convexity property. Even the mostly used “system optimization” function is in general not convex as we will show by a simple illustrative example (See Braess example in subsection 5.1).

### 3.3.2 Compromise Between Nash and MO

Assume that we have a concrete multi-leader tolling problem such that a Nash equilibrium exists. Can we find a "better" toll vector? Stakeholders can improve the system welfare (by solving MO) without deteriorating their individual utilities with reference to Nash outcome. On the other hand, if side payments are allowed, stakeholders will be better off cooperating or solving the MO. A possible model is given below:

Given the actors Nash equilibrium flow pattern \( (\bar{v}, d) \), the grand coalition game is given by (see MO program - Eqn 3):

\[
\min_{v, d} Z = \left( \sum_{k \in K} C_k(v) - \sum_{w \in W} \int B_w(\xi) d\xi \right)
\]

subject to

\[
\sum_{d} \quad FeC_{ED}
\]

\[
C_k(v) - \frac{1}{\lambda} \sum_{w \in W} \int B_w(\xi) d\xi \leq C_k(\bar{v}) - \frac{1}{\lambda} \sum_{w \in W} \int B_w(\xi) d\xi \quad \forall k \in K
\]

The objective maximizes the system’s economic benefit. The first two constraints are as explained before, the third constraint makes sure that no actor is worse off than in the Nash outcome. As the Nash flow pattern \( (\bar{v}, d) \) is a feasible solution for the above problem, it is always profitable for the stakeholders to agree on the solution \( (v, d) \) of the above program (see the case study). Note that given this solution \( (v, d) \), a corresponding first best pricing toll has to be chosen to make this solution also to a UE. If extra constraints on the tolls \( \theta \) are present, then a second-best pricing approach can be used in the same way. Omitting the last constraint, we can use side payments to assure each player his Nash outcome, and even more, additional benefits. This is always true since the total utility is optimal in the MO (Eqn 3).
4 OPTIMAL NASH INDUCING MECHANISM

We have shown so far that for the multi-leader model in Section 3, the existence of a NE cannot be guaranteed. In this section we therefore design a mechanism which induces a NE and even more returns the system optimal strategy as the optimal strategy for each actor. For this model we will assume that there exists a ‘grand leader (GL)’ who has authority over all other leaders (by adding one more uppermost level in Figure 1). Look at him as the central (or federal) government. His sole objective will be to ensure (Pareto) optimal social welfare of the entire system. Since competition may lead to tolls that deteriorate the social welfare, and since it is not clear if there is a profit sharing rule that leaves grand coalition as the only stable coalition among the actors (the core of the game), we develop a mechanism that achieves efficient and desirable global outcome irrespective of what the actors do. This mechanism aligns the objective of each actor with that of the GL. Thus, actors with once conflicting interests, now indirectly pursue common (GL) interest. To achieve this goal, the mechanism uses taxing scheme to simultaneously induce a pure NE and cooperative behaviour among actors, thus, yielding tolls that are optimal for the system.

We assume that the total revenue generated from the taxing scheme just as the tolls (by the stakeholders) are invested back into the system. We also assume that the actors’ utility functions are known to the GL. The tax can be seen as what an actor pays on the utility he enjoys for taking part in road pricing.

Note that for any solution \( \bar{v} \) of the models below, we can always choose a first-best pricing toll which ensures that \( \bar{v} \) is UE.

4.1 Mathematical Formulation of the Mechanism

4.1.1 Grand Leader’s Problem

Models described in this section will be on fixed demand \( d \) (the elastic case is straightforward). The GL problem is a multi-objective (grand coalition) optimization problem that searches for a flow pattern minimizing the entire system cost. The formulation is as follows (cf Eqn 3):

\[
\min_v Z(v) = \sum_{k \in K} C_k(v) \quad s.t \quad \begin{align*}
    v &= \Lambda f \\
    \Gamma f &= d \\
    \lambda &= \psi \\
    f &\geq 0 \quad \bar{\rho}
\end{align*}
\]  (12)

The constraints are the flow feasibility constraints and \( \{\psi \in \mathbb{R}^{|A|}, \lambda \in \mathbb{R}^{|W|}, \bar{\rho} \in \mathbb{R}^{|R|}\} \) are the KKT multipliers associated with the constraints. Let \( L \) be the Lagrangian and \( \bar{v} \) the solution to (12), then, there exists \( (\bar{\psi}, \bar{\lambda}, \bar{\rho}) \) such that the following KKT optimality conditions hold:

\[
L = \sum_{k \in K} C_k(v) + (\Lambda f - v)^T \psi + (d - \Gamma f)^T \lambda - f^T \bar{\rho}
\]

\[
\nabla_v L = \sum_{k \in K} \nabla C_k(\bar{v}) - \psi = 0 \Rightarrow \frac{d}{dv} \sum_{k \in K} C_k'(v_a) - \psi_a = 0 \quad \forall a \in A
\]  (13)

\[
\nabla_f L = \Lambda^T \psi - \Gamma^T \lambda - \bar{\rho} = 0 \Rightarrow \sum_{a \in A} \psi_a \delta_{ar} - \bar{\lambda}_a - \bar{\rho}_r = 0 \quad \forall r \in R_w, \forall w \in W
\]  (14)

\[
f^T \bar{\rho} = 0, \quad \bar{\rho} \geq 0, \Rightarrow \bar{\rho}_r f_r = 0 \quad \forall r \in R
\]  (15)

Equation (15) is called complementarity equation.

4.1.2 Stakeholder’s (or Actor’s) Problem

Having shown that NE does not exist in general, we discuss a mechanism where the GL chooses appropriate taxes \( x^k, k \in K \) that forces the game into a NE. The GL thus penalizes (taxes) \( k^{th} \) actor by \( v^T x^k \), where \( v^T \) is the transpose vector of link flows and \( x^k \in \mathbb{R}^{|A|} \) is a leader specific constant vector.

Now for fixed tax \( x^k \) each of the stakeholders \( k \in K \) solves the following optimization problem:

\[
\min_v Z_k(v) = C_k(v) + v^T x^k \quad s.t \quad \begin{align*}
    v &= \Lambda f \\
    \Gamma f &= d \\
    \lambda &= \psi \\
    f &\geq 0 \quad \rho
\end{align*}
\]  (16)

\( x^k \) is the constant vector for leader \( k \).
Let $L$ be the Lagrangian and $\tilde{v}$ the solution to (16), then, with $(\psi, \lambda, \rho)$, the following KKT conditions hold:

$$
L = C_k(v) + v^T x^k + (\Lambda f - v)^T \psi + (d^* - \Gamma f)^T \lambda + - f^T \rho
$$

$$
\nabla_v L = \nabla C_k(\tilde{v}) + x^k = 0 \implies \frac{d}{dv_a} C_k(\tilde{v}_a) + x^k_a - \psi_a = 0 \quad \forall a \in A
$$

$$
\nabla_f L = \Lambda^T \psi - \Gamma^T \lambda - \rho = 0 \implies \sum_{a \in A} \psi_a \delta_{ar} - \lambda_w - \rho_r = 0 \quad \forall r \in R_w, \forall w \in W
$$

$$
f^T \rho = 0, \rho \geq 0 \implies \rho_f f_r \quad \forall r \in R
$$

Observe that the only difference between the GL’s and the stakeholder’s KKT conditions is in Equations (13) & (17). Now, the GL can choose taxes $\bar{x}$ & (17) using the conditions above, we first state as for the first best tolls in (5), there exist infinitely many values for $x^k$ in the taxing schemes $\nu^T x^k$ (other than Eqns 20 & 21) that induce optimal Nash. Using the KKT optimality conditions above, we first state the following corollary

**Corollary 3:** If $\tilde{v}$ is the optimal flow vector for actor $k \in K$, then, the following holds (the proof is analogous to one seen in [15, 16] for Wardrop’s equilibrium):

$$
\sum_{a \in A} \left( \frac{d}{dv_a} C_k(v_a) \right)_{v_a = \tilde{v}_a} + x^k_a \delta_{ar} = \lambda_w + \rho_r \geq \lambda_w \quad \forall r \in R_w, \forall w \in W
$$

$$
\sum_{a \in A} \left( \frac{d}{dv_a} C_k(v_a) \right)_{v_a = \tilde{v}_a} + x^k_a \tilde{v}_a = \sum_{a \in A} \lambda_w d_w
$$

condensed to

$$
\Lambda^T \left( \nabla C_k(v)_{v = \tilde{v}} + x^k \right) \geq 1^T \lambda
$$

$$
\left( \nabla C_k(v)_{v = \tilde{v}} + x^k \right)^T \tilde{v} = d^T \lambda
$$

**Remark:** Observe from Equation (17) that a taxing scheme defined by the function $\nu^T x^k$ with

$$
x^k = \sum_{i \in K \setminus k} \nabla C_i(v)_{v = \tilde{v}}
$$

where $\tilde{v}$ is as defined in grand leader’s problem, is also an optimal Nash inducing scheme.

### 4.2 Flexible Taxing Scheme

Notice that the taxing mechanism above is analogous to first-best pricing tolling mechanism of the stakeholders on road users. So Equation (20) could be called first best pricing taxes. Thus, in the same way as for the first best tolls in (5), there exist infinitely many values for $x^k$ in the taxing schemes $\nu^T x^k$ (other than Eqns 20 & 21) that induce optimal Nash. Using the KKT optimality conditions above, we first state the following corollary

**Corollary 3:** If $\tilde{v}$ is the optimal flow vector for actor $k \in K$, then, the following holds (the proof is analogous to one seen in [15, 16] for Wardrop’s equilibrium):
for some $\lambda \geq 0$. The first line of Equation (22) states that each leader $k \in K$ would want each road user to follow the route that minimizes his (user’s) travel cost with respect to his (actor’s) objective function. The second line balances the network travel cost (w.r.t. $k$’s objective function) The following result on first-best taxes is analogous to corollary 1.

**Corollary 4:** Suppose $\bar{v}$ solves the GL’s problem, then, any taxing scheme $x^T \bar{x}^k$ such that $\bar{x}^k$ satisfies the following linear conditions is an optimal Nash inducing scheme on leader $k \in K$:

$$
\begin{align*}
\Lambda^T \left( \nabla C_k(v) \big|_{v=\bar{v}} + \bar{x}^k \right) &\geq \Gamma^T \lambda \\
\left( \nabla C_k(v) \big|_{v=\bar{v}} + \bar{x}^k \right)^T \bar{v} & = d^T \lambda
\end{align*}
$$

(24)

for some $\lambda \geq 0$.

**Proof:** The proof follows from Equation (23).

**Remarks**

1. Equations (20) & (21) directly satisfy condition (24).

2. With this flexible taxing scheme, the grand leader only needs to know the objective $C_k(v)$ of actor $k$ to determine $\bar{x}^k$. This mechanism can be compared with the social (usual) taxing scheme where taxes depend on income. Also, any of the stakeholders can pull out of the road pricing scheme without altering the model.

### 4.3 Secondary Objectives on the Taxing Scheme

Equation (24) suggests that we can set secondary objectives on these taxes. The following has intuitive meaning:

- Minimizing the total tax levied on each actor by solving the following linear system:

$$
\begin{align*}
\min_{\bar{x}^k} \bar{v}^T \bar{x}^k & \quad s.t. \quad \Lambda^T \left( \nabla C_k(v) + \bar{x}^k \right) \geq \Gamma^T \lambda \\
\left( \nabla C_k(v) + \bar{x}^k \right)^T \bar{v} & = d^T \lambda
\end{align*}
$$

(25)

where $\bar{v}$ is the GL desired link flow vector. For fairness, the GL may want to levy a flat tax on all stakeholders e.g $\bar{v}^T x^k = M$.

### 4.4 Coalition Among Leaders Under the Mechanism

**A1:** With the taxing scheme described, there does not exist a coalition in which any of the leaders is better off than in the induced Nash scenario.

**Proof:** Suppose such coalition exists, say with a feasible flow vector $\tilde{v}$ in which actor $k \in K$ is better off than in the Nash scenario, then, it simply contradicts the already established fact that the induced Nash flow vector $\tilde{v} \neq \bar{v}$ is the optimal (idle) flow vector for all leaders under the taxing scheme. Hence, such a coalition does not exist.

In fact, for an arbitrary coalition of two leaders $k$ and $m$:

$$
\begin{align*}
\tilde{C}_k(v) & = C_k(v) + v^T \bar{x}^k \\
\tilde{C}_m(v) & = C_m(v) + v^T x^m
\end{align*}
$$

where

$$
\bar{x}^k = \sum_{l \in K \setminus k} \nabla C_l(v) \big|_{v=\bar{v}}, \quad x^m = \sum_{l \in K \setminus m} \nabla C_l(v) \big|_{v=\bar{v}}
$$

as given in Equation (20) and $\bar{v}$ is the GL solution (see (12)). After coalition, their objective function is

$$
\tilde{C}_k(v) + \tilde{C}_m(v) = C_k(v) + C_m(v) + v^T (x^k + x^m)
$$

(26)

Given that $\bar{v} \in V$ minimizes Equation (26), then, *KKT* conditions for the minimization problem differs from those of stakeholder’s problem (Eqn 16) only in $\nabla L$ which is given by:

$$
\nabla L = \nabla C_k(v) \big|_{v=\bar{v}} + \nabla C_m(v) \big|_{v=\bar{v}} + \sum_{l \in K \setminus k} \nabla C_l(v) \big|_{v=\bar{v}} + \sum_{l \in K \setminus m} \nabla C_l(v) \big|_{v=\bar{v}} - \psi = 0
$$

(27)
Where $\tilde{v}$ is the GL’s optimal flow pattern. See that $\tilde{v} = \tilde{v}$ is a feasible solution for Equation (27), hence optimal. Hence, for $\tilde{v} = \tilde{v}$, Equation (27) becomes

$$2 \left( \sum_{l \in K} C_l(v) \right)_{v = \tilde{v}} - \psi = 0 \quad (28)$$

and since $\psi$ exists for the GL’s problem, there exists $\psi = 2\psi$ such that Equation (28) holds (see Eqn 13).

In the taxing scheme described above, we assumed that we can toll all links without bounds. This is the so called first best pricing scheme. In then next sections, we discuss the taxing mechanism with toll constraints/bounds. When tolls are not allowed on some links (second best pricing scheme), we face even a harder problem.

### 4.5 Optimal Nash Inducing Scheme for Second Best Pricing

#### 4.5.1 Unbounded Tolls

Here, we will see that the taxing scheme is also applicable when extra conditions on tolls are present and the first-best tolls are not no longer feasible.

#### 4.5.2 Grand Leader’s Problem

Suppose, we have the toll constraints $\theta_a \geq 0 \forall a \in A$, $\theta_d = 0 \forall a \in Y$, where $Y \subseteq A$. We reformulate system (12) as a bi-level optimization problem MPEC (see Eqn 4):

$$\min_{v, \theta, \lambda, k \in K} \sum_{k \in K} C_k(v) \quad s.t.$$

$$\Lambda^T (\beta_t(v) + \theta) \geq \Gamma^T \lambda$$

$$(\beta_t(v) + \theta)_v = (d^*)^T \lambda$$

$\theta_a \geq 0 \forall a \in A$

$\theta_d = 0 \forall a \in Y$

$\lambda \geq 0$

$v \in V$

The objective minimizes the system cost. The first two constraints are the user equilibrium conditions, and the last three contain the toll and the flow feasibility conditions (see Eqn 1).

#### 4.5.3 Stakeholder’s Problem

Each actor $k \in K$, instead of Eqn 16, now solves the following MPEC (see system 8):

$$\min_{v, \theta, \lambda} Z_k(v) = C_k(v) + v^T x_k \quad s.t.$$

$$\Lambda^T [\beta_t(v) + (\theta^k + \sum_{k \in K} \bar{\theta}^i)] \geq \Gamma^T \lambda$$

$$[\beta_t(v) + (\theta^k + \sum_{k \in K} \bar{\theta}^i)]_v = (d^*)^T \lambda$$

$\theta^k \geq 0 \forall a \in A$

$\theta^d = 0 \forall a \in Y$

$\lambda \geq 0$

$v \in V$

If we compare the KKT conditions of systems (29) and (30), then as in subsection 4.1, we have the following:

• let $\tilde{v}$ be the solution of program (29), then, if the GL chooses taxes $x^k$ as in (20), then the $\tilde{v}$ is also optimal for all stakeholders problem (30).

In fact, there is no problem arising from the extra conditions on tolls since system (30) holds for all $k \in K$, and this means that $\sum_{k \in K} \theta^k = 0 \forall a \in Y$. Since $\sum_{k \in K} \theta^d = \theta_d \forall a \in A$, it then means that system (30) satisfies/captures the toll constraint in the GL’s problem (29).

**Remarks**

1. The GL’s estimated link toll vector $\bar{\theta}$ is a valid Nash toll vector for the actors (recall the optimal Nash inducing scheme). One possible optimal toll vector for the actors is $\theta^d = \tilde{\theta}$ and $\theta^d = 0 \forall l \in K \setminus k$ assuming that actor $k$ makes the first move (i.e., actor $k$ is player 1). These links tolls are not unique in general, but, a Nash toll vector $\theta^d \forall k \in K$ is optimal for the actors if the cumulative toll $\sum_{k \in K} \theta^d = \theta_d \forall a \in A$ solves the GL’s problem (29).
2. Also other constraints on the tolls can be considered, for example, upper bounds \( \theta_a \leq \Phi_a \forall a \epsilon A \); with \( \Phi_a \epsilon \mathbb{R}_+ \). In this case, and for equity reasons, we have to assume each stakeholder has the link toll bound: \( \Phi_a = \frac{\theta_a}{K} \).

**General Remark**

The optimal inducing mechanism can also be used to check the (selfish) behaviour of:

1. malicious nodes in car to car communication where cars exchange data/information within a limited time frame.
2. local authorities tolling separate regions of the network.
3. energy producers in the energy markets liberalization problem.
4. agents in the principal-agents model.
5. Internet providers in the providers-subscribers Internet price setting problem.
6. employees that have flexibility on the number of workdays.

## 5 NUMERICAL EXAMPLES

### 5.1 The Braess Example

We use a well known example to show that even the total network travel time (objective), in general, may not be convex in the tolls (strategy set). Such a draw back is enough to ruin the existence of Nash equilibrium in the road pricing game. The yellow label is the unique link identity, numbering the links from 1 to 5. The other label is the cost a user encounters on using the link. The fixed demand from node \( a \) to \( d \) is 1. \( \theta_i \) represents the toll on link \( i \). For two classes of tolls, namely; \( \theta_3 \leq \theta_1 \) and \( \theta_3 \geq \theta_1 \) we have derived the following user equilibrated flows \( v_i \) on the links:

\[
\begin{align*}
    v_1 &= \begin{cases} 
    \theta_1 - \theta_1 & \text{if } \theta_1 - \theta_1 < 0.5 - 0.5 \theta_1 \\
    0.5 - 0.5 \theta_1 & \text{otherwise}
    \end{cases} \\
    v_2 &= \begin{cases} 
    1 - (\theta_1 - \theta_1) & \text{if } \theta_1 - \theta_1 < 0.5 - 0.5 \theta_1 \\
    0.5 + 0.5 \theta_1 & \text{otherwise}
    \end{cases} \\
    v_3 &= \begin{cases} 
    1 + \theta_1 - 2 \theta_1 & \text{if } \theta_1 - \theta_1 < 0.5 - 0.5 \theta_1 \\
    0 & \text{otherwise}
    \end{cases} \\
    v_4 &= \begin{cases} 
    1 - \theta_1 & \text{if } \theta_1 - \theta_1 < 0.5 - 0.5 \theta_1 \\
    0.5 - 0.5 \theta_1 & \text{otherwise}
    \end{cases} \\
    v_5 &= \begin{cases} 
    \theta_1 & \text{if } \theta_1 - \theta_1 < 0.5 - 0.5 \theta_1 \\
    0.5 + 0.5 \theta_1 & \text{otherwise}
    \end{cases}
\end{align*}
\]

\[
v^T(v) = \begin{cases} 
    2 \theta_1^2 + 2 \theta_1^3 & \text{if } \theta_1 - \theta_1 < 0.5 - 0.5 \theta_1 \\
    0.5 \theta_1^2 + 1.5 & \text{otherwise}
    \end{cases}
\]

By computing the user equilibrium (depending on \( \theta_1, \theta_3 \)), we see that the corresponding travel time function \( v^T(v) \) is not convex in the strategy set \( \{ \theta_1, \theta_3 \} \) (see the term \(-2 \theta_1 \theta_3 \) in \( v^T(v) \)). The graph shows how the tolls \( \theta_1, \theta_3 \) vary with the travel time function \( v^T(v) \). It shows that \( v^T(v) \) is not convex in the strategy \( \{ \theta_1, \theta_3 \} \). One can take the table of tolls above to mean a road pricing game between two distinct stakeholders who have interests on link 1 and 3 respectively. Suppose a stakeholder’s (which does not necessarily need to be the two distinct stakeholders) objective is to minimize the network travel time \( v^T(v) \), then in general, Nash equilibrium may not exist since \( v^T(v) \) is not convex in \( \{ \theta_1, \theta_3 \} \) even when other players objectives are convex in their strategy sets.
5.2 Five-Node Network Example

5.2.1 Link Attributes and Input

We will use a five-node network to illustrate the models developed in this paper. We demonstrate first-best scheme. For the second-best scheme we only need additional toll constraints.

FIGURE 2 The Five-Node Network with Eight Links (1,2,...,8)

Emission factors are from the CAR-model [21], emission costs is arbitrary, and noise costs are from [22]. The value of time (VOT) used is from [23]. The injury costs used are arbitrary. An optimization software AIMMS is used to solve the programs. All objectives include the user time cost \( t(v) \) as part of their social cost since users always perceive this cost in the lower level. We have used the following cost functions:

**System Travel Time Cost:**

\[
C_t(v) = \sum_{a \in A} \beta v_a l_a(v_a) + \sum_{a \in A} \beta v_a T_{ff} a \left(1 + \eta \left(\frac{\nu_a}{\nu_l}\right)^\phi\right);
\]

the so called Bureau for Public Roads (BPR) function, where
- \( T_{ff} a \) - free flow travel time on link \( a \),
- \( v_a \) - total flow on link \( a \),
- \( \nu_l \) - design standard of link \( a \), and
- \( \eta \) and \( \phi \) - BPR scaling parameters.

We will use \( \eta = 4, \phi = 0.15 \) and \( \beta \) (value of time - VOT) = €0.1671667/minute, see table 1a for other parameters.

**Emission Cost:**

\[
C_e(v) = \sum_{a \in A} \nu_a \rho a l_a(v_a) = \sum_{a \in A} \nu_a \left(\alpha_a \rho a l_a + \beta l_a(v_a)\right);\]

where
- \( \alpha_a \) - emission factor for link \( a \) (depending on the emission type and the vehicle speed on link \( a \) in g/vehicle-kilometre).
- \( l_a \) - length of link \( a \). In this case study, we only consider two emission types; \( NO_x \) and \( PM_{10} \).

See table 1c for the emission factor \( \rho a \) and lA for the emission costs \( \alpha_a \).

**Noise Cost:**

\[
C_n(v) = \sum_{a \in A} n_a(v_a) = \sum_{a \in A} \gamma \left[A + B\log\left(\frac{\nu_a}{\nu_l}\right) + 10\log\left(\frac{\nu_a}{\nu_l}\right)\right] l_a + v_a \beta l_a(v_a);\]

where
- \( A \) and \( B \) in \( dB(A) \) - vehicle specific constants as given in [24].
- \( v_a \) and \( v_0(v_{ref}) \) - the average and reference speed of vehicles on link \( a \) respectively.
- \( l_a \) - number of households along link \( a \).

We will use \( A = 69.4 dB(A), B = 27.6 dB(A) \) and \( v_0(v_{ref}) = 80 km/hr \), see table 1a for \( l_a \) and 1b for the monetary conversion parameter \( \gamma \).

**Infrastructure (Pavement) Cost:**

\[
C_i(v) = \sum_{a \in A} i_a(v_a) = \sum_{a \in A} \nu_a \left(\tau_a \left(\frac{H_a}{l_a}\right) l_a + \beta l_a(v_a)\right);\]

where
- \( \tau_a \) - load equivalence factor (LEF) that measures the amount of pavement deterioration produced by each vehicle on link \( a \).
- \( H_a \) - initial cost for the infrastructure per kilometre.
- \( J_a \) - design standard of link \( a \) measured by the design number of equivalent axle load (ESAL) repetitions.
- \( \frac{H_a}{l_a} \) - unit investment cost per ESAL-kilometre. The higher the design standard of an infrastructure, the smaller the factor \( \left(\frac{H_a}{l_a}\right) \), meaning that infrastructure with a high design standard are the most cost-effective way to handle high traffic volumes [25].

We will use \( \frac{H_a}{l_a} = 1 \). \( \tau_a \) is given in table 1a as cost of damage to infrastructure/link in €/vehicle-kilometre.

**Safety Cost:**

\[
C_s(v) = \sum_{a \in A} s_a(v_a) = \sum_{a \in A} \rho \tau_a E_a + v_a t_a(v_a) = \sum_{a \in A} \nu_a \left(\rho \tau_a l_a + \beta l_a(v_a)\right);\]

where
- \( \tau_a \) - risk factor for link \( a \), measured in number of injury crashes/vehicle-kilometre.
- \( E_a = l_a * v_a \) - measure of level of exposure on link \( a \).
We will set cost of one injury $\rho = \varepsilon 300/injury$.
Recall that the parameters ($\beta, \alpha, \gamma, \rho$) are the monetary conversion parameters described earlier in section 2.1.

### TABLE 1 Network Attributes

<table>
<thead>
<tr>
<th>Link</th>
<th>Length(km)</th>
<th>Free Speed(km/hr)</th>
<th>Capacity</th>
<th># of Households living around the links</th>
<th>Air Emission Cost for NOx(€/gram)</th>
<th>Air Emission Cost for PM10(€/gram)</th>
<th>Damage to infrastructure(€/veh-km)</th>
<th>Safety factor (jury per Veh-km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>100</td>
<td>420</td>
<td>420</td>
<td>1400</td>
<td>1000</td>
<td>2.0044</td>
<td>0.008</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>100</td>
<td>420</td>
<td>420</td>
<td>1400</td>
<td>1000</td>
<td>2.0044</td>
<td>0.008</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>100</td>
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<td>1000</td>
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<td>420</td>
<td>420</td>
<td>1400</td>
<td>1000</td>
<td>2.0044</td>
<td>0.008</td>
</tr>
</tbody>
</table>

**Cost of noise per household as measured from road traffic (€/year in 2003 price scale)**

<table>
<thead>
<tr>
<th>dB(A)</th>
<th>noise</th>
<th>55 - 65</th>
<th>66 - 75</th>
<th>&gt;75</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>22</td>
<td>45</td>
<td>65</td>
<td>80</td>
</tr>
</tbody>
</table>

**Speed (km/hr)** | Emission Factor (g/km/veh) | NOx | PM10 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5&lt;50</td>
<td>0.40</td>
<td>0.06</td>
<td></td>
</tr>
<tr>
<td>&gt;=50</td>
<td>0.48</td>
<td>0.059</td>
<td></td>
</tr>
<tr>
<td>&lt;=65</td>
<td>0.22</td>
<td>0.065</td>
<td></td>
</tr>
<tr>
<td>&gt;65</td>
<td>0.13</td>
<td>0.041</td>
<td></td>
</tr>
</tbody>
</table>

We define the following inverse demand (benefit) function for the OD pair $w = (a - e)$:

$$B(d_w) = 800 - \frac{d_w}{2}$$  \hspace{1cm} (31)

where $d_w$ is the variable OD demand for the $w^{th}$ OD.

#### 5.2.2 Results

Table 2 below shows the ideal link, path flows and the objective welfare when various objectives are singly optimized (as in Eqn 1) and in an aggregated multi-objective - MO (as in Eqn (32)) form. Of course this ideal link flows will never be achieved in practice (assuming an actor controls one objective) since the objectives are conflicting. In the table, UE displays the Wardropian equilibrium or a toll free network. Table 2b corresponds to the (non-unique) path flows as a result of table 2a. Table 2c (see the last column) displays the effect of single objective optimization on the system welfare and other objectives.

The table displays how single objective optimization can adversely affect other objectives (see negative entries) and the system. The social welfare is maximal when the objectives are optimized in an aggregated form (see Eqn(3)). The ideal welfare (bold diagonal entries) remain Pareto optimal for single objective optimizations and MO. The system/social welfare(SW) or economic benefit is defined as in Equations (3) & (32).

$$\begin{align*}
SW &= \sum_{a \in A} d_a \int_0^c B_a(\zeta) d\zeta - \sum_{a \in A} \left( b_{\nu_a l_a} + a_{\nu_a x_{l_a}} + 2 \gamma \left[ A + B \log \left( \frac{v_a}{v_0} \right) + 10 \log \left( \frac{v_a}{v_0} \right) \right] h_a + \zeta_{l_a} \frac{H_a}{l_a} + \rho_{\nu_a x_{l_a}} \right) (32)
\end{align*}$$
TABLE 2 Link Flows, Path Flows and Welfare Results

<table>
<thead>
<tr>
<th>Links/Objectives --&gt;</th>
<th>UE</th>
<th>Travel Time</th>
<th>Emission</th>
<th>Noise</th>
<th>Safety</th>
<th>Infrastructure</th>
<th>MO</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>524.50</td>
<td>533.43</td>
<td>517.65</td>
<td>526.61</td>
<td>457.94</td>
<td>532.69</td>
<td>277.45</td>
</tr>
<tr>
<td>2</td>
<td>500.05</td>
<td>428.15</td>
<td>368.67</td>
<td>423.75</td>
<td>0.00</td>
<td>428.33</td>
<td>302.14</td>
</tr>
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<td>3</td>
<td>565.70</td>
<td>512.95</td>
<td>0.00</td>
<td>520.58</td>
<td>529.90</td>
<td>513.17</td>
<td>450.84</td>
</tr>
<tr>
<td>4</td>
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<td>93.03</td>
<td>115.94</td>
<td>84.78</td>
<td>457.94</td>
<td>92.29</td>
<td>277.45</td>
</tr>
<tr>
<td>5</td>
<td>0.00</td>
<td>16.70</td>
<td>184.04</td>
<td>3.67</td>
<td>0.00</td>
<td>16.13</td>
<td>0.00</td>
</tr>
<tr>
<td>6</td>
<td>482.15</td>
<td>440.41</td>
<td>401.71</td>
<td>441.83</td>
<td>0.00</td>
<td>440.40</td>
<td>0.00</td>
</tr>
<tr>
<td>7</td>
<td>542.40</td>
<td>504.48</td>
<td>300.57</td>
<td>504.86</td>
<td>457.94</td>
<td>504.49</td>
<td>579.59</td>
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<tr>
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<td>565.70</td>
<td>529.64</td>
<td>184.04</td>
<td>524.25</td>
<td>529.90</td>
<td>529.30</td>
<td>450.84</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Paths (r) /Path flows (f_i) --&gt;</th>
<th>UE</th>
<th>Travel Time</th>
<th>Emission</th>
<th>Noise</th>
<th>Safety</th>
<th>Infrastructure</th>
<th>MO</th>
</tr>
</thead>
<tbody>
<tr>
<td>a→b→e</td>
<td>482.15</td>
<td>440.41</td>
<td>401.71</td>
<td>441.83</td>
<td>0.00</td>
<td>440.40</td>
<td>0.00</td>
</tr>
<tr>
<td>a→b→c→d→e</td>
<td>42.35</td>
<td>93.03</td>
<td>74.76</td>
<td>84.78</td>
<td>457.94</td>
<td>92.29</td>
<td>277.45</td>
</tr>
<tr>
<td>a→b→c→d</td>
<td>0.00</td>
<td>0.00</td>
<td>41.18</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>a→c→d→e</td>
<td>500.05</td>
<td>411.46</td>
<td>225.83</td>
<td>420.04</td>
<td>0.00</td>
<td>412.24</td>
<td>302.14</td>
</tr>
<tr>
<td>a→d→e</td>
<td>565.70</td>
<td>512.95</td>
<td>0.00</td>
<td>520.58</td>
<td>529.90</td>
<td>513.17</td>
<td>450.84</td>
</tr>
<tr>
<td>Total OD demand</td>
<td>1590.25</td>
<td>1474.53</td>
<td>886.32</td>
<td>1470.94</td>
<td>987.84</td>
<td>1474.19</td>
<td>1030.43</td>
</tr>
</tbody>
</table>

The objective values in welfare table is derived from the input functions for those objectives (see also Eqn (1)).

5.3 Cooperative and Non-cooperative Leaders’ Game

For clarity, we will only consider three actors whose interests are respectively to maximize the societal welfare (Eqn 32) considering only system travel time cost, emission cost and safety cost respectively. We will denote these actors by “r”, “e” and “s” respectively. We assume non-negative link tolls.

5.3.1 The Cooperative Game

For comparison reasons, we have shown the result of a cooperative game among the three actors (see table 3a). They now solve the multi-objective models of Equation (3) using (32) (aggregating the three costs). Recall that taxing scheme described will also lead to this cooperative outcome.

5.3.2 One Shot Game

Here, without any knowledge of other actors’ toll vectors, the actors set tolls in one shot to optimize their individual objectives (see 1). Each actor sets his ideal toll that would lead to his optimal flow in single leader game.

The cumulative link toll vector resulted (see Table 3b) in a welfare of €177,350 which is 44% less than cooperative outcome €316,674 of the same game. This reveals that the societal welfare from actions of uncoordinated actors can leave the network or the market far from optimal. Note that the tolls are in general not unique.
5.3.3 Nash Game

Here actors iteratively solve their individual MPECs (system 8). The game terminates (NE) when no actor can increase his objective by changing his current toll vector given that other leaders’ strategies are fixed.

We solve the Nash game using the NIRA-3 [26]. NIRA-3 is a MATLAB package that uses the Nikaido-Isoda function and relaxation algorithm to find unique Nash equilibria in infinite games. Conditions on tolls ensure the existence of Nash equilibrium. For more on the NIRA-3 see [26].

Using $\texttt{alphamethod} = 0.5$, $\texttt{precision} = [1e{-3}, 1e{-3}]$, and $\texttt{TolCon} = \texttt{TolFun} = \texttt{TolX} = 1e{-3}$, $\texttt{Linktoll bound for eachactor} = [0, \infty]$EUR it took NIRA-3 approximately 2 minutes in 70 iterations to find the NE (see Table 3).

The Nash game (table 3c) shows some improvements of €129,561 (73%) on the social welfare with regard to the one-shot game (which is the worst case scenario of the Nash game). Iterative process of the Nash game tends to inform actors about other actors’ objectives, sort of a coordinated game. In some limited sense, actors, during the iterative process, indirectly solve a multi-objective problem [27, 28]. See also that the cooperative game improves the social welfare of this game by €9,763 (3%).

5.3.4 Users Interest

Users interests in the upper level (see figure 1) may be in form of an alternative (lower) toll vector that achieves the same Nash flow pattern for the stakeholders. For the game with fixed demand, there may
exist a lower toll vector for the same Nash flow. For elastic demand, users can gain much in toll reduction by slightly deteriorating the actors’ utilities, because for elastic demand, the total toll revenue is the same for all toll patterns [16]. This slight deterioration is easily covered by the gain in toll reduction so that the actors are not left worse off than in Nash game.

In the Nash game of table 3c, users can ask for 22% (€68,164) deterioration of the total actors’ utility for a whooping reduction of 73% (€220,000) in the toll 'burden' (total toll revenue). See in Table 3d that the utility of player “t” appreciates while those of “e” and “s” depreciate under this new scheme. The users offset the €68,164 loss in the actors’ utility from the €220,000 gain in toll reduction; guaranteeing each actor the Nash outcome and possibly increasing their utilities by some $\varepsilon > 0$ for stability reasons. Notice the “user friendly” link toll (total) pattern as compared to the previous games. The following formulation can be used for this problem:

$$\text{max (the actors' total utility – the actors' total utility with respect to Nash)}$$

$$\text{s.t} - \text{the total toll reduction (gain) } \geq \text{€220,000}$$

$$\text{-the total toll gain } \geq \text{the total utility gap}$$

$$\text{-the equilibrium (Equation 4) and feasibility conditions (Equation 1)}$$

5.3.5 Compromise Between Nash and MO

See from table 3 that each stakeholder is better off in the cooperative game than in the Nash game. This is a coincidence for this game structure. What is true is that, the total utility for all actors is always bigger in the cooperative game than in the Nash game, see table 4 for example. So, for side payment games, there is always a payment scheme (e.g. Egaiterian rule that distributes excess (w.r.t. Nash) utility) that induces grand coalition among players.

5.3.6 Non-Existence of Nash Equilibrium

In this part, we demonstrate the non-existence of NE with unbounded tolls (subsection 3.3). Suppose actors toll the network in a sequential manner (“t” ==> “e” ==> “s” ==> “t” ==> “e” ==> “s”... and so on), we will show that they will always achieve their ideal flow (and hence their optimal objective value) in each move (see corollary 2). Let us represent the stakeholders’ action by $\theta^k_i$ which is the toll vector of actor $k$ in the $i^{th}$ move. We also denote by $\bar{\theta}^k$ the ideal toll vector for actor $k$ (see one-shot game of Table 3).

Let player “t” be the first to toll the network, he thus uses his ideal link toll vector

$$\theta^t_1 = \bar{\theta}^t = (28.77, 29.34, 29.76, 0.00, 0.00, 29.76, 29.58, 28.86)^T$$

then for player “e” to achieve his ideal flow, he will now toll (see Eqn 11)

$$\theta^e_1 = \bar{\theta}^e - \theta^t_1 = (147.17, 147.82, 148.55, 0.78, 0.00, 148.55, 148.73, 148.54)^T$$

the next move will be player “s”, and to achieve his ideal flow vector, his action is as follows:

$$\theta^s_1 = \bar{\theta}^s - (\theta^t_1 + \theta^e_1) = (-54.82, -52.69, -57.13, -0.78, 153.82, -54.25, -58.31, -55.33)^T$$

the next will be player “t” again, his ideal flow now is achieved by the following toll vector, and so on

<table>
<thead>
<tr>
<th>Links</th>
<th>$\theta^t_1$</th>
<th>$\theta^e_1$</th>
<th>$\theta^s_1$</th>
<th>$\theta^t_2$</th>
<th>$\theta^e_2$</th>
<th>$\theta^s_2$</th>
<th>$\theta^t_3$</th>
<th>$\theta^e_3$</th>
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<th>Tolls</th>
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<td>1</td>
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<td>441.51</td>
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<td>-108.27</td>
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</table>

illustrating our statement that Nash equilibrium does not exist with unbounded tolls. The same argument holds for fixed demand even with non-negative toll restriction (corollary 2).
6 CONCLUDING REMARKS AND FUTURE RESEARCH

6.1 Contributions and Conclusion

We presented a game theoretical approach to solve the multi-objective road pricing including externalities other than congestion. Due to political and equity reasons, various stakeholders and/or regions may partake in toll setting. Since stakeholders may have objectives that do not align with each other, we studied in this paper, the existence of Nash equilibrium among these actors. We showed that in more practical settings, we do not expect NE to exist. We also represented the road users’ interest in the upper level and showed by means of example that such an idea can lead to better acceptance of road pricing schemes. Since actors cannot be forced to form the grand coalition, we developed a mechanism that simultaneously induces a pure NE and cooperative behaviour among actors, thus, yielding optimal tolls for the system.

The cooperative result has many advantages which include: 1) it protects local citizens from the negative effects of other jurisdiction’s pricing policies, 2) it eliminates the finance externality which reduces demand for local roads from non local residents and hurts profit. We also saw that the model developed in this paper is applicable in many interesting instances.

6.2 Research Extensions and Recommendations

Since the models used in this paper centered around classical optimization formulations, the number of variables can grow uncontrollably large with large networks. This calls for an efficient optimization heuristic algorithms for large networks. Since flat link charge alone as described in this paper may not fully attribute an externality to a car, for example, emission costs, we will extend our models to a kilometer charge, which will then take care of the (current) taxes on gasoline, diesel an petrol. Furthermore, a working paper is extending the models of this paper to multimodal settings and time dependent models, and further investigating under which specific conditions a (unique) Nash equilibrium exists among non-cooperative stakeholders.

References


