Graphs and Combinatorics (2012) 28:57–75
DOI 10.1007/s00373-011-1090-6

SURVEY

How Many Conjectures Can You Stand? A Survey

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Received: 1 September 2011 / Revised: 17 September 2011 / Published online: 9 October 2011

Abstract We survey results and open problems in hamiltonian graph theory centered around two conjectures of the 1980s that are still open: every 4-connected claw-free graph (line graph) is hamiltonian. These conjectures have lead to a wealth of interesting concepts, techniques, results and equivalent conjectures.

Keywords Hamiltonian graph · Hamilton-connected · Claw-free graph · Line graph · Cubic graph · Dominating closed trail · Dominating cycle · Collapsible graph · Supereulerian graph · Snark · Cyclically 4-edge-connected · Essentially 4-edge-connected · Closure · Contractible graph

Mathematics Subject Classification (2000) 05C45 · 05C38 · 05C35

1 Introduction

Before we are going to introduce the necessary terminology for understanding the sequel, let us start by presenting the two conjectures that will play the main role throughout our exposition.

Zdeněk Ryjáček and Petr Vrána were supported by project 1M0545 and Research Plan MSM 4977751301 of the Czech Ministry of Education.

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Most of the results in this survey paper are inspired by the following two conjectures that were tossed in the 1980s, and later appeared in the cited papers. The first conjecture is due to Matthews and Sumner [50].

Conjecture 1 *Every 4-connected claw-free graph is hamiltonian.*

The second conjecture due to Thomassen was posed in [60], but was already mentioned in 1981 on page 12 of [6], and also appeared in [1].

Conjecture 2 *Every 4-connected line graph is hamiltonian.*

The above two highly related conjectures and their relationship to other open problems and results have been the subject of a number of specialized small scale workshops between 1996 and 2011 in Enschede, Nečtiny (twice), Hannover, Hájek and Domažlice (twice). In order to make the material available to a larger community we decided to compose this survey paper that contains most of the relevant material related to these intriguing open conjectures.

The presented material involves—apart from line graphs and claw-free graphs—cubic graphs, snarks, and concepts like Hamilton cycles, Hamilton-connectedness, dominating closed trails (circuits), and dominating cycles, and techniques involving closures, collapsible graphs, and edge-disjoint spanning trees.

The paper is organized as follows. We first continue in the next section by explaining the necessary terminology to understand the above statements and their relationship. Next we will introduce the tools that show that the two conjectures are in fact equivalent, and we analyze what the statement of the latter conjecture would mean for the root graph of the line graph. Then we will present a sequence of seemingly weaker but equivalent conjectures, and of seemingly stronger but equivalent conjectures. We finish with a survey of some of the existing partial solutions to the conjectures, and discuss how far we are from either proving or refuting the conjectures.

2 Basic Terminology and Concepts

All graphs in this survey are finite, undirected and loopless, and the majority is simple (in some results we allow multiple edges). We refer to [10] for standard terminology and notation.

We denote a (simple) graph G as $G = (V, E)$, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set.

Adopting the terminology of [10], a graph is called *hamiltonian* if it contains a *Hamilton cycle*, i.e., a cycle containing all its vertices, i.e., a connected spanning 2-regular subgraph.

If H is a graph, then the *line graph of* H, denoted by $L(H)$, is the graph on vertex set $E(H)$ in which two vertices in $L(H)$ are adjacent if and only if their corresponding edges in H share an end vertex (with a straightforward extension in case of multiple edges).

A graph G is a *line graph* if it is isomorphic to $L(H)$ for some graph H.

Which graphs are line graphs (of simple graphs) and which are not? This question was answered by a forbidden subgraph characterization due to Beineke [5].

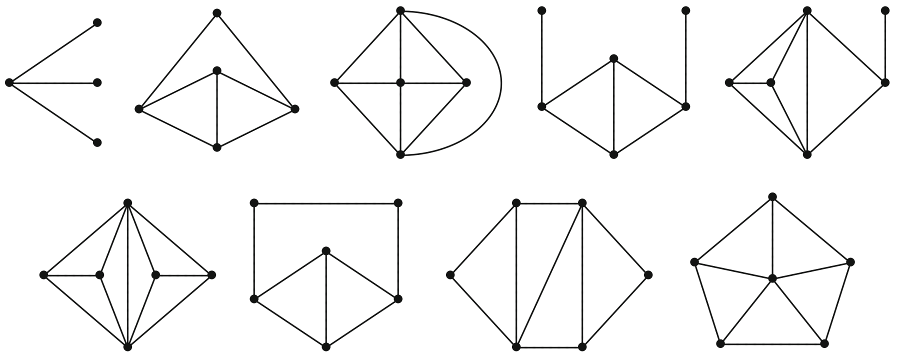

Fig. 1 The nine forbidden subgraphs for line graphs of simple graphs

Theorem 3 *A graph G is a line graph if and only if G does not contain a copy of any of the graphs of Fig.* 1 *as an* induced *subgraph.*

Let G be a graph and let S be a nonempty subset of $V(G)$. Then the subgraph of G *induced by* S, denoted by $G[S]$, is the graph with vertex set S, and all edges of G with both end vertices in S. H is an *induced subgraph* of G if it is induced in G by some subset of $V(G)$. G is H*-free* if H is not an induced subgraph of G. In particular, a graph G is *claw-free* if G does not contain a copy of the *claw* $K_{1,3}$ as an induced subgraph. Direct inspection of Beineke's result shows that every line graph is claw-free.

3 A Handful of Conjectures and More

Since line graphs are claw-free, Conjecture 1 is stronger than Conjecture 2. Or are they equivalent? (A question Herbert Fleischner posed during the EIDMA workshop on Hamiltonicity of 2-tough graphs, Hotel Hölterhof, Enschede, November 19-24, 1996 [8].)

To answer the question affirmatively, Zdeněk Ryjáček introduced a *closure concept* for claw-free graphs at the same workshop which was published in [53]. It is based on adding edges without destroying the (non)hamiltonicity (similar to the Bondy–Chvátal closure [9] for graphs with nonadjacent pairs with high degree sums).

The edges are added by looking at a vertex v and the subgraph of G induced by $N(v)$: the *neighborhood* of v.

If $G[N(v)]$ is connected and not a complete graph, all edges are added to turn $G[N(v)]$ into a complete graph.

This procedure is repeated in the new graph, etc., until it is impossible to add any more edges. By the following theorem due to Ryjáček [53], the closure $cl(G)$ we obtain this way is a well-defined graph.

Theorem 4 *Let G be a claw-free graph. Then*

– *the closure* $cl(G)$ *is uniquely determined,*
– $cl(G)$ *is hamiltonian if and only if G is hamiltonian,*
– $cl(G)$ *is the line graph of a triangle-free graph.*

The above theorem also shows that Conjectures 1 and 2 are equivalent. Moreover, it gives the opportunity to translate questions on hamiltonicity in claw-free graph to questions on hamiltonicity in line graphs, and results on line graphs to results on the more general class of claw-free graphs. We come back to this later when we discuss partial solutions to the two conjectures. Variants on the above closure technique and extensions are discussed in [18].

Here we follow the line of reasoning by turning our attention to what the statements of the conjectures entail for the *root graph* of the line graph.

Whenever we consider a line graph G, we can identify a graph H such that $G = L(H)$. If G is connected this H is unique, except for $G = K_3$: then H can be K_3 or $K_{1,3}$ (this is different for multigraphs, where we could also have three parallel edges, or two parallel edges and one additional incident edge; and there are other pairs of connected multigraphs with isomorphic line graphs). If we restrict ourselves to simple graphs and take $K_{1,3}$ in this exceptional case, we can talk of a unique graph H as the root graph of the connected line graph G isomorphic to $L(H)$. What is the counterpart in H of a Hamilton cycle in G? A *closed trail* (sometimes referred to as a *circuit* in the literature) is a connected *eulerian* subgraph, i.e., a connected subgraph in which all degrees are even. A *dominating closed trail* (DCT for short) is a closed trail T such that every edge has at least one end vertex on T. Note that this notion of domination is not equivalent to the usual notion of domination meaning that every vertex not on the trail has a neighbor on the trail; in our case of a DCT T in a graph H, the graph $H - V(T)$ is edgeless. Also note that a DCT might consist of only one vertex (in case the graph H is a star; then $L(H)$ is a complete graph).

There is an intimate relationship between DCTs in H and Hamilton cycles in $L(H)$, a result due to Harary and Nash-Williams [30] that is known since the 1960s.

Theorem 5 *Let H be a graph with at least three edges. Then $L(H)$ is hamiltonian if and only if H contains a DCT.*

What is the counterpart in H of 4-connectivity in $L(H)$? Note that 4-edge-connectivity is not the right answer, because edge-cuts in H that consist of all edges incident to a single vertex v of H do not correspond to vertex-cuts in $L(H)$ if $H - v$ has at most one component containing edges. A graph H is *essentially* 4*-edge-connected* if it contains no edge-cut R such that $|R| < 4$ and at least two components of $H - R$ contain an edge. It is not difficult to check that $L(H)$ is 4-connected if and only if H is essentially 4-edge-connected. The previous results and observations imply that the following conjecture is equivalent to Conjectures 1 and 2.

Conjecture 6 *Every essentially* 4*-edge-connected graph has a DCT.*

If H is *cubic*, i.e., 3-regular, then a DCT becomes a *dominating cycle* (abbreviated DC). H is *cyclically* 4*-edge-connected* if H contains no edge-cut R such that $|R| < 4$ and at least two components of $H - R$ contain a cycle. It is not difficult to show that a cubic graph is essentially 4-edge-connected if and only if it is cyclically 4-edge-connected. Hence the following conjecture due to Ash and Jackson [2] is a specialization of Conjecture 6 to cubic graphs.

Conjecture 7 *Every cyclically* 4*-edge-connected cubic graph has a DC.*

Plummer [52] observed that Conjecture 7 is equivalent to the following two specializations of Conjecture 1.

Conjecture 8 *Every* 4*-connected* 4*-regular claw-free graph is hamiltonian.*

Conjecture 9 *Every* 4*-connected* 4*-regular claw-free graph in which each vertex lies on exactly two triangles is hamiltonian.*

Fleischner and Jackson [25] proved that Conjecture 7 is in fact also equivalent to the others. First note that one can transform an essentially 4-edge-connected graph into one with minimum degree at least three by first deleting the vertices with degree 1, and then replacing the paths with internal vertices with degree 2 by edges (*suppressing* vertices with degree 2). The main ingredient in their proof is a nice trick to replace vertices with degree more than 3 in the obtained graph by cycles without affecting the essentially 4-edge-connectivity.

Let H be an essentially 4-edge-connected graph of minimum degree $\delta(H) \geq 3$ and let $v \in V(H)$ be of degree $d(v) \geq 4$. Delete v and add a cycle on $d(v)$ new vertices, and join the new vertices to the original neighbors of v by a perfect matching. The resulting graph is called a *cubic inflation* of H at v. It is not unique, since it depends on the choice of the matching edges joining the new vertices to the original neighbors of v. Fleischner and Jackson [25] proved that by a suitable choice of these edges, some cubic inflation of H at v results in an essentially 4-edge-connected graph. By repeating this procedure, the resulting graph will eventually be cubic and still essentially (and hence cyclically) 4-edge-connected.

Before we continue with imposing further restrictions on the cubic graphs under consideration, we would like to mention the following two related conjectures that have been stated in [25] and are due to Jaeger and Bondy, respectively.

Conjecture 10 *Every cyclically* 4*-edge-connected cubic graph* G *has a cycle* C *such that* $G - V(C)$ *is acyclic.*

Conjecture 11 *Every cyclically* 4*-edge-connected cubic graph* G *on* n *vertices has a cycle of length at least* cn*, for some constant* c *with* $0 < c < 1$.

It is obvious that Conjecture 7 implies Conjecture 10, and it is not difficult to show that Conjecture 10 implies Conjecture 11. We are not aware of any attempts to establishing the equivalence of these conjectures, and we leave it as an open problem.

A further restriction to cyclically 4-edge-connected cubic graphs that are *not 3-edge-colorable*, is due to Fleischner [24] who posed the following conjecture.

Conjecture 12 *Every cyclically* 4*-edge-connected cubic graph that is not* 3*-edge-colorable has a DC.*

Kochol [39] proved that Conjecture 12 is equivalent to the others, by a constructive approach. By assuming a counterexample to Conjecture 7 and using this as a black box building block, he was able to construct a counterexample to Conjecture 12, using an auxiliary gadget that is almost cubic and not 3-edge-colorable. We skip the details.

For our final restriction on the cyclically 4-edge-connected cubic graphs under consideration, we now turn to snarks. In this paper a *snark* is defined as a cyclically 4-edge-connected cubic graph of girth at least 5 that is not 3-edge-colorable. Here the *girth* of a graph G is the length, i.e., the number of edges or vertices, of a shortest cycle in G. In the literature one can find several variants on this definition where either the restriction on the cyclically edge-connectivity or on the girth or on both are relaxed. Snarks turn up as the 'difficult' objects in many open problems in graph theory, including conjectures on double cycle covers and nowhere zero flows. These are beyond the scope of this survey. We refer to the books of Zhang [66] and [67] for more details and background.

The next conjecture has appeared independently at different places.

Conjecture 13 *Every snark has a dominating cycle.*

Conjecture 13 is also equivalent to the others, as shown in [13], using the constructive approach together with the concept of *contractible subgraphs*. We will explain some of the key ingredients here but refer to [13] for more details. The first step in the proof of the equivalence is based on a refinement of a technique introduced in [56].

In [56], the notion of A*-contractible* graphs is introduced. For a graph H and a subgraph F of H, $H|_F$ denotes the graph obtained from H by contracting F to a single vertex and adding some new vertices and edges in order to keep the same number of edges. This is done by identifying the vertices of F as one new vertex v_F, replacing the edges between vertices of F and vertices of $V(H) \setminus V(F)$ by the same number of edges between v_F and the adjacent vertices of $V(H) \setminus V(F)$, and by replacing the created loops (i.e., one for each edge of F) by *pendant* edges, i.e., edges incident with v_F and one other newly added incident vertex of degree 1. Note that $H|_F$ may contain multiple edges but has the same number of edges as H. A vertex of F is a vertex of *attachment* if it has a neighbor in $V(H) \setminus V(F)$. The set of vertices of attachment of F with respect to H is denoted by $A_H(F)$.

For a subset $X \subset V(H)$, and a partition $\mathscr{A}$ of X into subsets, $E(\mathscr{A})$ denotes the set of all edges a_1a_2 (not necessarily in H) such that a_1, a_2 are in the same element (i.e., the same equivalence class) of $\mathscr{A}$. Now $H^{\mathscr{A}}$ denotes the graph with vertex set $V(H^{\mathscr{A}}) = V(H)$ and edge set $E(H^{\mathscr{A}}) = E(H) \cup E(\mathscr{A})$ (where $E(H)$ and $E(\mathscr{A})$ are considered to be disjoint, so if $e_1 = a_1a_2 \in E(H)$ and $e_2 = a_1a_2 \in E(\mathscr{A})$, then e_1 and e_2 are parallel edges in $H^{\mathscr{A}}$).

Let F be a graph and $A \subset V(F)$. Then F is A*-contractible*, if for every even subset $X \subset A$ (i.e., with $|X|$ even) and for every partition $\mathscr{A}$ of X into two-element subsets, the graph $F^{\mathscr{A}}$ has a DCT containing all vertices of A and all edges of $E(\mathscr{A})$. Note that the case $X = \emptyset$ implies that an A-contractible graph has a DCT containing all vertices of A.

The importance of A-contractible graphs lies in the fact proved in [56] that a connected graph F is A-contractible if and only if, for any H such that $F \subset H$ and $A_H(F) = A$, H has a DCT if and only if $H|_F$ has a DCT. In fact, the authors of [56] proved the stronger result that the (extended) contraction (as defined above) of an A-contractible subgraph of a graph H does not affect the maximum number of edges dominated by a closed trail in H. Note that this number corresponds to the length of a longest cycle in $L(H)$.

In [13], the following slightly weaker notion of a *weakly A-contractible* graph plays an essential role. The difference with the above notion is that only nonempty even subsets $X \subset A$ are required to have the above property. This means that a weakly A-contractible graph is not required to have a DCT containing all vertices of A. Using this weaker notion, one of the key auxiliary results proved in [13] yields that for a 2-connected cubic graph H with a weakly $A_H(F)$-contractible subgraph F of H, H has a DC if and only if $H|_F$ has a DCT. This obviously imposes structural restrictions on possible minimal counterexamples to the conjectures on the existence of a DC in certain cubic graphs. This is combined in [13] with a second step in which it is shown that replacing a subgraph of a cubic graph does not affect the (non)existence of a DC if certain *compatible mappings* are respected. Without going into the technical details of explaining what these mappings entail, this enables the replacement of 4-cycles in a possible counterexample to Conjecture 7 in order to construct a counterexample with girth at least 5 (Note that the only cyclically 4-edge-connected cubic graph with triangles is K_4). This is then further combined in [13] with techniques that were previously used in [39] in order to construct a snark without a DC under the assumption of a counter example to Conjecture 7.

We like to bring the following two conjectures that were posed in [13] to the reader's attention. The first of these two conjectures was shown to be equivalent to the other conjectures.

Conjecture 14 *Every cyclically* 4*-edge-connected cubic graph contains a weakly contractible subgraph F with* $\delta(F) = 2$.

The following statement, also posed as a conjecture in [13], implies the above, but we do not know whether it is equivalent to the above conjecture.

Conjecture 15 *Every cyclically* 4*-edge-connected cubic graph G contains a weakly contractible subgraph F with* $|A_G(F)| \geq 4$.

To date Conjecture 13 is the seemingly weakest conjecture on the existence of a DC in certain cubic graphs that is equivalent to Conjectures 1 and 2. All snarks up to 36 vertices were tested for the existence of a DC by Brinkmann et al. [11]. Due to the role snarks play in other areas we would like to pose the following two open questions.

– Is there a link to conjectures on *Double Cycle Covers*?
– Is there a link to conjectures on *Nowhere-Zero Flows*?

Taking a slightly different approach, we continue with presenting some other seemingly weaker conjectures. Kochol [40] proved equivalence with seemingly weaker versions, using a concept called *sublinear defect*. As an example, he proved that Conjecture 2 is equivalent to the following conjecture.

Conjecture 16 *There are sublinear functions* $f_1(n)$ *and* $f_2(n)$ *such that every* 4*-connected line graph G of order n contains* $\leq f_1(n)$ *paths that cover* $\geq n - f_2(n)$ *vertices of G.*

Similar techniques were introduced and applied in [3] to obtain equivalent versions of the 2-tough conjecture, and in [4] successfully applied with suitable small gadgets to obtain counterexamples to the 2-tough conjecture. Although the 2-tough conjecture restricted to claw-free graphs is equivalent to Conjecture 1, it is beyond the scope of this survey. We refer the reader to [12] for more details. Inspired by these techniques, independently of [39] it has been shown in [14] that Conjectures 1 and 2 are equivalent to seemingly weaker conjectures in which the conclusion is replaced by a conclusion similar to the one in Conjecture 16. We use the term *r-path-factor* for a spanning subgraph consisting of at most r paths. A *2-factor* is a set of vertex-disjoint cycles that together contain all the vertices of the graph, i.e., a 2-regular spanning subgraph.

Theorem 17 *Let $k \geq 2$ be an integer, and let $f(n)$ be a function of n with the property that $\lim_{n\to\infty} \frac{f(n)}{n} = 0$. Then the following statements are equivalent.*

(1) Every k-connected claw-free graph is hamiltonian.
(2) Every k-connected claw-free graph on n vertices has an $f(n)$-path-factor.
(3) Every k-connected claw-free graph on n vertices has a 2-factor with at most $f(n)$ components.
(4) Every k-connected claw-free graph on n vertices has a spanning tree with at most $f(n)$ vertices of degree one.
(5) Every k-connected claw-free graph on n vertices has a path of length at least $n - f(n)$.

The key ingredient for proving the above equivalences is the auxiliary result proved in [14] that the existence of a k-connected nonhamiltonian claw-free graph G on n vertices implies the existence of such a graph G^* on at most $2n - 2$ vertices that contains a *k-clique*, i.e., a set of k mutually adjacent vertices. This result enables the construction of k-connected claw-free graphs on at most $(2r + 1)(2n - 2)$ vertices without an r-path-factor, assuming that there is a k-connected nonhamiltonian claw-free graph G on n vertices, by simply taking $2r + 1$ vertex-disjoint copies of G^* and adding all edges between the k-clique vertices of all the copies.

By results in [32], where it has been shown that a claw-free graph G has an r-path-factor if and only if $cl(G)$ has an r-path-factor, and in [55], where it has been shown that a claw-free graph G has a 2-factor with at most r components if and only if $cl(G)$ has such a 2-factor, the equivalence of statements (1), (2) and (3) in the above theorem also holds for line graphs.

In this section we have presented a sequence of gradually seemingly weaker conjectures that turned out to be equivalent. In the next section we are going to present some seemingly stronger conjectures.

4 Seemingly Stronger Versions for Cubic Graphs

Fouquet and Thuillier [27] considered a seemingly stronger version than the Ash-Jackson-Conjecture (Conjecture 7). Although the next conjecture is equivalent to Conjecture 7, the conclusion is stronger in the sense that it requires a DC containing any two given disjoint edges, as follows.

Conjecture 18 *In a cyclically 4-edge-connected cubic graph any two disjoint edges are on a DC.*

Establishing equivalent conjectures with stronger conclusions might help in an attempt to refute the conjectures. The above equivalence was extended by Fleischner and Kochol [26] by requiring a DC through any two given edges.

Conjecture 19 *In a cyclically* 4*-edge-connected cubic graph any two edges are on a DC.*

Brinkmann et al. [11] have verified Conjecture 19 for all not 3-edge-colorable cyclically 4-edge-connected cubic graphs with girth at least 4 up to 34 vertices, and for all snarks on 36 vertices.

There are several further equivalent versions involving other subgraphs of cubic graphs, like Conjecture 14. We present two others here without going too much into the technical details. Interested readers are invited to consult the sources [43] and [45], respectively. We need some additional terminology. Let H be a graph with minimum degree $\delta(H) = 2$ and suppose that the set $V_2(H)$ of all vertices with degree 2 in H has four elements. We say that H is *$V_2(H)$-dominated* if the graph $H + \{e_1, e_2\}$ arising from H after adding two new edges $e_1 = xy$ and $e_2 = wz$ (possibly creating multiple edges) such that $\{x, y, w, z\} = V_2(H)$ has a dominating closed trail containing e_1 and e_2. We say that H is *strongly $V_2(H)$-dominated* if H is $V_2(H)$-dominated and moreover the graph $H+e$ obtained from H by adding the new edge e has a dominating closed trail containing e for any newly added edge $e = uv$ for $\{u, v\} \subset V_2(H)$.

The following two conjectures appeared in [43] and [45], respectively.

Conjecture 20 *Any subgraph H of an essentially* 4*-edge-connected cubic graph with $\delta(H) = 2$ and $|V_2(H)| = 4$ is $V_2(H)$-dominated.*

Conjecture 21 *Any subgraph H of an essentially* 4*-edge-connected cubic graph with $\delta(H) = 2$ and $|V_2(H)| = 4$ is strongly $V_2(H)$-dominated.*

We now turn to seemingly stronger versions than Conjecture 2 for line graphs. Adopting the terminology of [10], a graph is called *Hamilton-connected* (sometimes called hamiltonian-connected in the literature) if it admits a Hamilton path between any two distinct given vertices. It is easy to check that any Hamilton-connected graph on at least 4 vertices is necessarily 3-connected.

Kužel and Xiong [46] established the equivalence of Conjecture 2 with the following conjecture.

Conjecture 22 *Every 4-connected line graph of a multigraph is Hamilton-connected.*

Ryjáček and Vrána [58] further extended the equivalence to claw-free graphs by proving that the following conjecture is equivalent to Conjecture 22.

Conjecture 23 *Every 4-connected claw-free graph is Hamilton-connected.*

One of the key ingredients in their equivalence proof is a result from [57] that extends the closure technique used in [53] to establish the equivalence of Conjectures 1 and 2. In this new version of the closure technique, the *2-closure*, edges are added to a noncomplete neighborhood in a claw-free graph G if this neighborhood

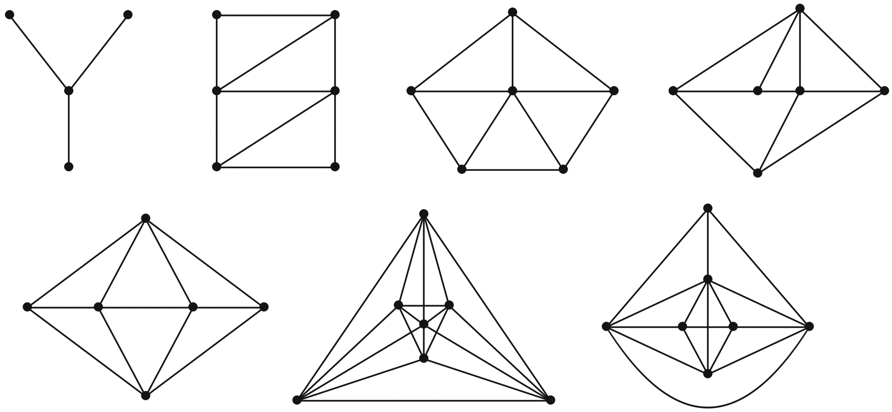

Fig. 2 The seven forbidden subgraphs for line graphs of multigraphs

induces a 2-connected subgraph instead of just a connected one. Then it is proved that the new graph G' is Hamilton-connected if and only if G is Hamilton-connected. We note here that it is not always true that a Hamilton path between two vertices u and v exists in G if and only if it exists in G'. Successively adding edges to a claw-free graph G according to this new version yields a unique graph denoted $cl_2(G)$. One of the serious difficulties in this approach is that the successive application of this new closure operation to 2-connected neighborhoods does not always result in a line graph (of a multigraph). One of the structures that can appear in $cl_2(G)$ is the *square* of a cycle, i.e., the graph obtained from a cycle by adding edges between nonadjacent vertices that have a common neighbor. The closure operation defined in [58] deals with these squares of cycles separately (by adding all the edges to turn them into complete graphs on the same vertex set) and defines an additional closure operation on *good walks* in the graph $cl_2(G)$ if it is not the square of a cycle. We will not explain the details involved in the handling of these good walks, but we conclude here with the statement that this extension guarantees that the resulting *multigraph closure* is a unique graph $cl_M(G)$, and that it is the line graph of a multigraph. Moreover, this new graph $cl_M(G)$ is Hamilton-connected if and only if the original graph G is Hamilton-connected. For convenience, we add the counterpart of Fig. 1 which shows the forbidden induced subgraphs of line graphs of multigraphs. These are illustrated in Fig. 2.

5 A Link to the P Versus NP Problem

At present the seemingly strongest version of the conjectures for line graphs is by Kužel, Ryjáček and Vrána [45].

Adopting the terminology of [45], a graph G is called *1-Hamilton-connected* if for any vertex x of G there is a Hamilton path in $G - x$ between any two vertices, and G is called *2-edge-Hamilton-connected* if the graph $G + X$ has a Hamilton cycle containing all edges of X for any $X \subseteq \{xy|x, y \in V(G)\}$ with $1 \leq |X| \leq 2$. It is easy to check that for both properties 4-connectedness is a necessary condition (except for complete graphs on at most 4 vertices).

Using the equivalence of Conjecture 2 and Conjecture 21, in [45] it is proved that the following conjecture is equivalent to Conjecture 2.

Conjecture 24 *Every* 4-*connected line graph of a multigraph is* 1-*Hamilton-connected (2-edge-Hamilton-connected).*

This version strongly suggests that Conjecture 2 (and all equivalent versions) might fail, for the following reasons. If the above conjecture is true, it implies that a line graph is 1-Hamilton-connected (2-edge-Hamilton-connected) if and only if it is 4-connected. It is well-known that the connectivity of a (line) graph can be determined in polynomial time. It is an NP-complete problem to decide whether a line graph is hamiltonian (see, e.g., [7]). It is not difficult to show that deciding whether a given graph is 1-Hamilton-connected is also NP-complete. It seems not unlikely that deciding whether a given graph is 1-Hamilton-connected remains NP-complete when restricted to line graphs. If one would be able to show this, however, it would imply that Conjecture 2 cannot be true, unless P=NP. In other words, the validity of Conjecture 2 would imply polynomiality of both 1-Hamilton-connectedness and 2-edge-Hamilton-connectedness in line graphs.

We add here as a side remark that, on the other hand, it is an easy exercise to show that a result of Sanders (see [59, p. 342]) implies that every 4-connected planar graph is 1-Hamilton-connected. Thus for a given planar graph one can decide in polynomial time whether it is 1-Hamilton-connected or not, whereas deciding whether a planar graph is hamiltonian is an NP-complete problem.

6 One Step Beyond

Very recently, the closure techniques of [57,58] have been strengthened and adapted to work for the stronger notion of 1-Hamilton-connectivity. In [44], the concept of multigraph closure is further strengthened in such a way that this adapted closure of a claw-free graph is the line graph of a multigraph with at most two triangles or at most one double edge. In [54], this is used to obtain a closure that turns a claw-free graph into a line graph of a multigraph while preserving the property of (not) being 1-Hamilton-connected. This yields the following currently seemingly strongest version of the conjectures.

Conjecture 25 *Every 4-connected claw-free graph is 1-Hamilton-connected.*

7 Positive Results Related to the Conjectures

The gap between the conjecture(s) and the positive results is narrowing, in the following sense. If we look at the connectivity conditions in Conjectures 1 and 2, then the first natural question is whether one can prove a theorem on hamiltonicity of claw-free graphs or line graphs if one imposes a stronger connectivity condition. The earliest result in this direction is due to Zhan [65] (and was independently proved by Jackson [33]).

Theorem 26 *Every 7-connected line graph (of a multigraph) is hamiltonian.*

In fact, Zhan proved the stronger result that such graphs are Hamilton-connected. For this purpose, he slightly generalized Theorem 5 to formulate an equivalent result on the existence of dominating trails between pairs of edges in the root graph H of a line graph $G = L(H)$ such that each edge of H is dominated by an internal vertex of the trail. He then used an approach that is typical for most of the results in this section. We will present some of the ingredients here, starting with a classic result on the existence of k edge-disjoint spanning trees due to Nash-Williams [51] and Tutte [61].

Theorem 27 *A graph G has k edge-disjoint spanning trees if and only if for every partition $\mathscr{P}$ of $V(G)$ we have $\varepsilon(\mathscr{P}) \geq k(|\mathscr{P}| - 1)$, where $\varepsilon(\mathscr{P})$ counts the number of edges of G joining distinct parts of $\mathscr{P}$.*

Kundu [42] observed that Theorem 27 has the following consequence.

Theorem 28 *Every k-edge-connected graph has at least $\lceil (k-1)/2 \rceil$ edge-disjoint spanning trees.*

The use of the existence of two edge-disjoint spanning trees for obtaining a spanning eulerian subgraph was observed by several researchers independently, and appeared in a paper by Jaeger [36].

Theorem 29 *Every graph with two edge-disjoint spanning trees has a spanning eulerian subgraph.*

The intuition behind this result is that the vertices of odd degree in one of the trees can be paired and connected by edge-disjoint paths in the other tree to form a spanning eulerian subgraph (a spanning closed trail).

Combining the above results, we immediately obtain the next corollary.

Corollary 30 *(i) Every 4-edge-connected graph has a spanning eulerian subgraph.*
(ii) Every 4-edge-connected graph has a hamiltonian line graph.

On the other hand, we know that Conjecture 2 is equivalent to the conjecture (see Conjecture 6) that every essentially 4-edge-connected graph has a hamiltonian line graph. At first sight the gap between Corollary 30(i) and Conjecture 6 does not look that large. Moreover in Corollary 30(i) we obtain a spanning eulerian subgraph, whereas we would only need a DCT, i.e., a dominating eulerian subgraph in order to prove Conjecture 6. Nevertheless Conjecture 6 and all the equivalent conjectures seem to be very hard. As a side remark and a possible approach to solving the conjectures, we would like to present another conjecture, that would clearly imply Conjecture 6, and was put up by Jackson [34]. It resembles the way one can prove that 4-connected planar graphs are hamiltonian by proving assertions on the existence of certain cycles (paths) in 2-connected planar graphs.

Conjecture 31 *Every 2-edge-connected graph G has an eulerian subgraph H with at least three edges such that each component of $G - V(H)$ is linked by at most three edges to H.*

Vrána [63] recently observed that Conjecture 31 is equivalent to Conjecture 2.

We continue with sketching the approach to proving Theorem 26 and similar results. Similarly to the way we have been proving the equivalence of many of the conjectures mentioned earlier, the first step is to consider the root graph of the line graph, and the equivalent property one has to establish, e.g., the existence of a DCT or of a trail between two given edges that internally dominates all edges of the root graph. In the next step the root graph is usually reduced by deleting the vertices with degree one (or with only one neighbor in the case of multigraphs) and suppressing the vertices with degree two. In the third step the degree and connectivity properties of the reduced graph are used to establish the existence of a spanning eulerian subgraph (or trail between two given edges). In this step the existence of two disjoint spanning trees (or something slightly more sophisticated) is usually the intermediate goal.

Theorem 26 has been extended to results on 6-connected line graphs with some additional conditions. The proof in [65] together with Theorem 4 immediately implies that every 6-connected claw-free graph G with $\delta(G) \geq 7$ is hamiltonian. More careful considerations show that the condition $\delta(G) \geq 7$ can be weakened to 'at most 33 vertices have degree 6' (Li [8]) or 'the vertices of degree 6 are independent' (Fan [8]). Further extensions to 6-connected line graphs with some additional conditions and the conclusion Hamilton-connected, but following basically the same method as in [65], can be found in [31]. Even further extensions can be found in [64], but they still need an additional condition bounding the number of vertices with degree 6 to at most 74 or the structure they induce to at most 8 disjoint K_4s (for 6-connected claw-free graphs to be hamiltonian) or bounding the number of vertices with degree 6 to at most 54 or the structure they induce to at most 5 disjoint K_4s (for 6-connected line graphs to be Hamilton-connected). The proofs in [64] use a similar approach as in the above sketch, but combined with a powerful reduction technique based on *collapsible* graphs introduced by Catlin [19]. Since this technique and its refinements play an important role in obtaining results on the existence of spanning closed trails and DCTs, we will give a brief outline of the basics involved. Before doing so, we first present the currently best connectivity result related to Conjectures 1 and 2 due to Kaiser and Vrána [37].

Theorem 32 *Every* 5*-connected claw-free graph with minimum degree at least* 6 *is Hamilton-connected.*

The proof of Theorem 32 is very technical and too complicated and long to present here. Basically, the proof is along the same lines as the proofs of the other results in this section. However, instead of finding two edge-disjoint spanning trees the authors use a far more sophisticated approach to find *quasitrees* with *tight complements* in *hypergraphs* associated with the root graphs. They apply this to prove that an essentially 5-edge-connected graph in which every edge has at least 6 neighboring edges contains a connected eulerian subgraph spanning all the vertices of degree at least 4. This suffices to prove Theorem 32 for line graphs and with the conclusion hamiltonian. Refinements of the techniques then show the validity of the more general statement. The authors state in their concluding section of [37] that it is conceivable that a further refinement in some parts of their analysis might improve the result a bit, perhaps even to all 5-connected line (claw-free) graphs. On the other hand, they believe that the 4-connected case would require major new ideas. For instance, the root graph H of a

4-connected line graph may be cubic, in which case it is not clear how to associate a suitable hypergraph with H in the first place.

To finish this section we give the basic definitions and results related to the technique of collapsible graphs. We refer to [20] for a survey on applications of the technique.

A graph is called *supereulerian* if it contains a spanning eulerian subgraph. A graph H is *collapsible* if for every even subset X of $V(H)$, H has a subgraph H_X such that $H - E(H_X)$ is connected and X is the set of odd degree vertices of H_X. As examples, it is easy to see that a cycle of length 3 (or an edge of multiplicity 2 in a multigraph) is a collapsible graph and it is not difficult to show that a graph containing two edge-disjoint spanning trees is a collapsible graph. But also many graphs that are only a few edges short of having two edge-disjoint spanning trees are collapsible (see, e.g., [21]). The importance of collapsible graphs is immediate from the following result proved by Catlin [19].

Theorem 33 *If H is a collapsible subgraph of a graph G, then G is supereulerian (collapsible) if and only if G/H is supereulerian (collapsible).*

Here G/H is the graph obtained from G by contracting all edges of H and removing all loops. The theorem gives a powerful reduction method for studying supereulerian graphs because one can contract any collapsible subgraph without affecting this property. It was shown in [19] that any (multi)graph G has a unique collection of maximal collapsible subgraphs, so contracting them yields a well-defined unique graph called the *reduction* of G. Apart from applications in the area of our survey, there are many applications of the above reduction method in the study of cycle double covers, nowhere-zero 4-flows, etc. These are beyond the scope of this survey.

Motivated by the idea to modify the above technique to the study of DCTs instead of spanning closed trails, Veldman [62] refined Catlin's technique by handling vertices of degree 1 and 2 in a special way (since degree 1 vertices cannot occur on any closed trail, and the two neighbors of a degree 2 vertex are on any DCT). This refinement can be described in the following way. For a simple graph H, let $D(H) = \{v \in V(H) | d_H(v) = 1, 2\}$. For an independent set X of $D(H)$, let $I_X(H)$ be the graph obtained from H by contracting one edge incident with each vertex of X. Veldman then defined H as X*-collapsible* if $I_X(H)$ is collapsible in the Catlin sense. Also this refined reduction technique is a powerful tool for studying hamiltonicity of line graphs, in particular for dense graphs. However, the main drawback of Catlin's and Veldman's techniques is that the search for maximal collapsible subgraphs is very difficult. In this context, a natural question is whether the claw-free closure concept can be strengthened by using line graph techniques or by combining them with closure techniques. A first attempt in this direction was done in [17], but the major work was done in [56], where it was shown that the reduction techniques of Catlin and Veldman can be reformulated in terms of a closure technique for line graphs. This closure technique might be more convenient to use since it avoids the necessity of a search for maximal collapsible subgraphs. It is based on the concept of A-contractible graphs that was introduced earlier. We refer to [56] for more details and to [18] for a survey on closure techniques (this survey does not contain the work of [56]).

8 Related Results with a Weaker Conclusion

First of all, if we drop the connectivity condition of the 2-regular spanning subgraph, we move from a Hamilton cycle to a 2-factor. Enomoto et al. [22] proved that every 2-*tough* graph contains a 2-factor. Since $2k$-connected claw-free graphs are k-tough by a result in [50], this implies the following.

Theorem 34 *Every 4-connected claw-free graph has a 2-factor.*

It does not seem easy to use this as a starting point to show that there is a 2-factor with only one component, although there are some results that give upper bounds on the number of components (see, e.g., [15,16,29]). These results are beyond the scope of this paper.

By Theorem 3.1 in Jackson and Wormald [35], every connected claw-free graph has a 2-*walk*, i.e., a closed walk which passes every vertex at most twice. Clearly, the edges of a 2-walk induce a connected factor with maximum degree at most 4. In [14] the following related result is proved.

Theorem 35 *Every* 4*-connected claw-free graph contains a connected factor which has degree two or four at each vertex.*

By the results of Kriesell [41] it is possible to prove the related result that between every pair of distinct vertices in a 4-connected line graph there exists a spanning trail which passes every vertex at most twice. As with the 2-factor result these results do not seem to help in finding a way to prove Conjectures 1 and 2, although they supply some supporting evidence in favor of the conjectures.

9 Related Results with Additional Conditions

We have already presented some results in which a connectivity condition is accompanied by another condition, e.g., Theorem 32. Another way of obtaining positive results related to the conjectures is by relaxing the 4-connectedness and adding something else. Many such results involve degree conditions and other neighborhood conditions. Such results have been surveyed in several papers (see, e.g., [12,23,28]). We do not want to discuss such conditions in this survey, but here is a connectivity-only result.

If we add an ‘essentially connectivity’ condition there is this result due to Lai et al. [48].

Theorem 36 *Every* 3*-connected, essentially* 11*-connected claw-free (line) graph is hamiltonian.*

The proof of Theorem 36 is based on the technique of collapsible graphs by Catlin applied to the graph obtained from the root graph of the line graph by deleting vertices with degree 1 and suppressing vertices with degree 2. We omit the details.

Recently, Kaiser and Vrána [38] were able to decrease the 11 to 9 in the above theorem. In their proof they use a slight modification of their proof approach to Theorem 32 in [37]. The proof is again based on quasitrees with tight complements in hypergraphs, but in the proof they have to work around quasitrees which contain *bad type leaves*. This can be done by suitably choosing the hyperedges of the associated hypergraph. We refer to [37] for the details.

Perhaps the 11 in Theorem 36 can be replaced by 5, which would be best possible (by the line graph of the Petersen graph in which the edges of a perfect matching are subdivided exactly once). An open question is how far we can decrease the 11 (or 9) by raising the 3 to a 4 in the theorem.

10 Restrictions on the Root Graph

Using the technique of collapsible graphs, Lai [47] proved the following partial affirmative answer to Conjecture 2 by restricting the root graph to the class of *planar* graphs, i.e., graphs that can be embedded in the plane in such a way that the edges only intersect in incident vertices.

Theorem 37 *Every 4-connected line graph of a planar graph is hamiltonian.*

Kriesell [41] proved a similar result on line graphs of claw-free (multi)graphs with the stronger conclusion of Hamilton-connectedness. In fact, he proved the following more general result.

Theorem 38 *Let G be a graph such that $L(G)$ is 4-connected and every vertex of degree 3 in G is on an edge of multiplicity at least 2 or on a triangle of G. Then $L(G)$ is Hamilton-connected.*

Lai, Shao and Zhan [49] did something similar for *quasi claw-free* graphs, i.e., in which every pair of vertices u and v at distance 2 has a common neighbor w the neighbors of which are in $N(u) \cup N(v) \cup \{u, v\}$.

Theorem 39 *Every 4-connected line graph of a quasi claw-free graph is Hamilton-connected.*

11 Conclusion

We presented many conjectures, most of which have been shown to be equivalent to the conjecture that 4-connected claw-free graphs are hamiltonian. We also presented several results that supply supporting evidence in favor of the conjectures, including the most recent result that 5-connected claw-free graphs with minimum degree at least 6 are Hamilton-connected. There are many other results on hamiltonian properties of sufficiently connected claw-free graphs, including many that have not been listed here. In most of the proofs of the results that are closely related to the open conjectures, closure techniques are used to restrict the statements to line graphs. Then the root graphs are considered and the aim is to find a (closed or open) trail (internally) dominating all edges. A common approach is the following. First the degree 1 vertices are deleted, then the degree 2 vertices are suppressed, and now one tries to show that the reduced graph has a suitable spanning (closed) trail. This is usually accomplished by applying the technique of finding two edge-disjoint spanning trees (or similar structures that yield suitable trails), or by the technique of collapsible subgraphs, or by advanced closure concepts. It seems that none of these techniques is capable of tackling the open conjectures. Does the latter conclusion suggest that the conjectures are all false? We now tend to believe that there might exist nonhamiltonian 4-connected claw-free graphs, but we have no strong opinion. It is our sincere hope that this survey will inspire new research into this intriguing and challenging field.

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