Stabilization of sandwich non-linear systems with low-and-high gain feedback design

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Abstract—In this paper, we consider the problems of semi-global and global internal stabilization of a class of sandwich systems consisting of two linear systems with a saturation element in between. We develop here low-and-high gain and scheduled low-and-high gain state feedback design methodologies to solve the posed stabilization problems.

I. INTRODUCTION

Physical systems are typically made up of interconnected subsystems, some of which are well-characterized as linear, and some of which are distinctly nonlinear. Many systems can therefore be described as an interconnection of separable linear and nonlinear parts. One common type of structure consists of a static nonlinearity sandwiched between two linear systems, as shown in Figure 1. We observe that such sandwich non-linear systems are extensive generalizations of linear systems subject to actuator saturation. Our focus in this paper is on sandwich non-linear systems where the static nonlinearity is a saturation element as shown in Figure 2. We develop here low-and-high gain and scheduled low-and-high gain state feedback design methodologies to stabilize such sandwich systems either semi-globally or globally. The developed methods are generalizations of classical low-and-high gain and scheduled low-and-high gain state feedback design methodologies which have been conceived and have been successfully used to stabilize linear systems subject to actuator saturation, to enhance their performance (see for instance, [1], [2], [3], [5], [6]).

In an earlier paper [13], we developed the necessary and sufficient conditions under which sandwich non-linear systems of the type in Figure 2 and their generalizations can be stabilized either semi-globally or globally. We also developed low-gain and generalized scheduled low-gain design methodologies for constructing appropriate stabilizing controllers. The philosophy in the previous work can be briefly sketched as follows: we designed a controller such that the saturation does not get activated after some finite time. Thereafter, the design methodology reduces to a simple low gain or scheduled low gain design. However, such design methods based on standard low-gain or scheduled low gain design methods are conservative as they are constructed in such a way that the control forces do not exceed a certain level in an arbitrary, a priori given, region of the state space in the semi-global case or the whole state space in the global case. Hence the saturation remains inactive. Therefore, such generalized low-gain design methods do not allow full utilization of the available control capacity. Design methods based on low-and-high gain feedback design are conceived to rectify the drawbacks of low-gain design methods, and can utilize the available control capacity fully. As such, they have been successfully used for control problems beyond stabilization, to enhance transient performance and to achieve robust stability and disturbance rejection [2], [3], [5].

It is prudent to first review previous research on sandwich systems such as depicted in Figures 1 and 2, which are special cases of so-called cascade systems consisting of linear systems whose output affects a nonlinear system. The research on such cascaded systems was initiated in [4] but has also been studied in for instance [7], [8]. Note that in our case the nonlinear system has a very special structure of an interconnection of a static nonlinearity with a linear system. Moreover, in these references the nonlinear system is assumed to be stable and the goal was to see whether the output of a stable linear system can affect the stability of cascaded system. The goal of this paper, being focused on

Fig. 1. Static nonlinearity sandwiched between two linear systems

Fig. 2. Single-layer sandwich system

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developing methods for designing stabilizing controllers, is inherently different.

Also, some other researchers have previously studied linear systems with sandwiched nonlinearities. The most recent activity in this area is the work of Tao and his coworkers [9], [10], [11], [12]. The main technique used in these papers is based on approximate inversion of nonlinearities. An example studied in these references is a deadzone, which is a right-invertible nonlinearity. By contrast, a saturation has a very limited range and cannot be inverted even approximately, except in a local region. The work of Tao et al. is therefore not applicable to the case of a saturation nonlinearity. To achieve our goal of semi-global and global stabilization, we need to face the saturation directly, by exploiting the structural properties of the given linear systems.

This paper is organized as follows: In section II, we formulate the semi-global and global stabilization problems and present the necessary and sufficient conditions for solvability which have been obtained in [13]. A generalized low-high gain feedback design methodology as well as its scheduled version are introduced and semi-globally and globally stabilizing controllers are constructed in Section III. Some illustrating examples are given in Section IV.

II. PROBLEM FORMULATION

Consider two linear systems, denoted by $L_1$ and $L_2$, and given by:

\begin{align}
L_1: \begin{cases}
\dot{x}(t) &= Ax(t) + Bu(t) \\
z(t) &= Cx(t),
\end{cases} \\
L_2: \quad \dot{\omega}(t) &= M\omega(t) + N\sigma(z(t)),
\end{align}

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $z(t) \in \mathbb{R}^q$ and $\omega \in \mathbb{R}^m$. Here $\sigma()$ denotes the standard saturation function defined as $\sigma(u) = [\sigma_1(u_1), \ldots, \sigma_1(u_q)]$ where $\sigma_1(s) = \text{sgn}(s) \min \{|s|, 1\}$.

**Problem 1** Consider the systems given by (1) and (2). The semi-global stabilization problem is said to be solvable if there exists for any compact set $\mathcal{W} \subset \mathbb{R}^{n+m}$, a state feedback control law $u = f(x, \omega)$ such that the equilibrium point $(0, 0)$ of the closed-loop system is asymptotically stable with $\mathcal{W}$ contained in its domain of attraction.

**Problem 2** Consider the systems given by (1) and (2). The global stabilization problem is said to be solvable if there exists a state feedback control law $u = f(x, \omega)$ such that the equilibrium point $(0, 0)$ of the closed-loop system is globally asymptotically stable.

The necessary and sufficient conditions for solvability of semi-global and global stabilization problems as formulated above have been established in [13], which are stated in the following theorem:

**Theorem 1** Consider the interconnection of the two systems given by (1) and (2). The semi-global and global stabilization problems, as formulated in Problems 1 and 2 respectively, are solvable if and only if.

1) All the eigenvalues of $M$ are in the closed left half plane.

2) The linearized cascade system is stabilizable, i.e. $(\hat{A}, \hat{B})$ is stabilizable, where

$$
\hat{A} = \begin{pmatrix} A & 0 \\ N & M \end{pmatrix} \quad \text{and} \quad \hat{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}.
$$

Moreover, the solution to the semi-global stabilization problem can be achieved by a linear state feedback law of the form $u = Fx + G\omega$.

A generalized low-gain feedback design methodology has been given in [13]. In subsequent sections, we introduce a different strategy for stabilization of sandwich non-linear systems, namely a generalized low-high gain feedback design methodology (for semi-global stabilization) and its scheduled version (for global stabilization) which are capable of enhancing the system performance such as robust stability and disturbance rejection.

III. SEMI-GLOBAL AND GLOBAL STABILIZATION OF SANDWICH NON-LINEAR SYSTEM USING LOW-HIGH-GAIN FEEDBACK CONTROLLER

A. Semi-global controller design

We first choose $F$ such that $A + BF$ is asymptotically stable and consider the system:

\begin{align}
\dot{x} &= (A + BF)x + Bv \\
z &= Cx
\end{align}

where $u = Fx + v$. We have

$$
z(t) = Ce^{(A+BF)t}x(0) + \int_0^t Ce^{(A+BF)(t-\tau)}Bv(\tau)\,d\tau = Ce^{(A+BF)t}x(0) + z_0(t)
$$

Since $A + BF$ is asymptotically stable, we know that there exists a $\delta$ such that

$$
||v(\tau)|| < \delta \quad \forall \tau > 0
$$

implies that $||z_0(t)|| < \frac{\delta}{2}$.

Next we consider the system,

\begin{align}
\begin{pmatrix}
\dot{x} \\
\dot{\omega}
\end{pmatrix} = \begin{pmatrix} A + BF & 0 \\ N & M \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \nu.
\end{align}

Our objective is, for any a priori given compact set $\mathcal{W}$, to find a stabilizing controller for the system (6) such that $\mathcal{W}$ is contained in its domain of attraction and $||v(\tau)|| < \delta$ for all $\tau > 0$.

Let $Q_2 > 0$ be a parameterized family of matrices which satisfies $\frac{Q_2}{2\nu} > 0$ for $\varepsilon > 0$ with $\lim_{\nu \to 0} Q_2 = 0$. In that case, there exists for any $\varepsilon > 0$ a $P_\varepsilon > 0$ satisfying

\begin{align}
\begin{pmatrix} A + BF & 0 \\ N & M \end{pmatrix}' P_\varepsilon + P_\varepsilon \begin{pmatrix} A + BF & 0 \\ N & M \end{pmatrix} - P_\varepsilon \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} P_\varepsilon + Q_\varepsilon = 0.
\end{align}

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We first show the following lemma.

**Lemma 1** Consider the system (6) and assume that the pair $(\tilde{A}, \tilde{B})$ as given by (3) is stabilizable and the eigenvalues of $M$ are in the closed left half plane. Then, for any a priori given compact set $W \in \mathbb{R}^{n+m}$, there exists an $\epsilon^*$ such that for any $0 < \epsilon < \epsilon^*$ and $\rho > 0$, the state feedback,

$$v = -\delta \sigma \left( \frac{1+\rho}{\delta} \right) \left( B \right)' \left( P \tilde{x} \right),$$

(8)

achieves asymptotic stability of the equilibrium point $\tilde{x} = 0$ where we denote by $\tilde{x}$ the state of the system (6). Moreover, for any initial condition in $W$, the constraint $\|v(t)\| \leq \delta$ does not get violated for any $t > 0$.

**Proof:** Note that condition 2 of Theorem 1 immediately implies the existence of a $P_\varepsilon > 0$ satisfying (7). Moreover, condition 1 immediately implies that $P_\varepsilon \rightarrow 0$ (9) as $\varepsilon \rightarrow 0$. Obviously, controller (8) satisfies $\|v\| < \delta$. It remains to show that such a controller achieves semi-global stabilization. Define $V(\tilde{x}) = \tilde{x}'P_\varepsilon \tilde{x}$. Let $c$ be defined as

$$c = \sup_{\tilde{x} \in W} \{ \tilde{x}'P_\varepsilon \tilde{x} \}. $$

There exists an $\epsilon^*$ such that for any $\epsilon \in (0, \epsilon^*)$, we have that $\tilde{x} \in \mathcal{L}_c(\varepsilon) = \{ \tilde{x} \mid \tilde{x}'P_\varepsilon \tilde{x} \leq c \}$, implies that $\|\tilde{v}\| \leq \delta$ where we denote

$$\tilde{v} = \left( B \right)' \left( P_\varepsilon \right) \tilde{x}. $$

Consider $\dot{V}$ along any trajectory,

$$\dot{V} = -\tilde{x}'Q_\varepsilon \tilde{x} - 2\delta \tilde{v}'[\sigma(\frac{1+\rho}{\delta} \tilde{v}) - \frac{1}{\delta} \tilde{v}]$$

$$= -\tilde{x}'Q_\varepsilon \tilde{x} - 2\delta \tilde{v}'[\sigma(\tilde{v}) - \sigma(\frac{1}{\delta} \tilde{v})].$$

We have $\dot{V} < 0$ for any $\rho > 0$. This completes the proof. ■

**Theorem 2** Consider the interconnection of the two systems given by (1) and (2) satisfying conditions 1 and 2 of Theorem 1. Let $F$ be such that $A + BF$ is asymptotically stable while $P_\varepsilon > 0$ is defined by (7). Define a state feedback law by

$$u = F\tilde{x} - \delta \sigma(\frac{1+\rho}{\delta}) \left( B \right)' \left( P_\varepsilon \right) \left( x \right)$$

(10)

Then, for any compact set of initial conditions $W \in \mathbb{R}^{n+m}$, there exists an $\epsilon^* > 0$ such that for all $\epsilon$ with $0 < \epsilon < \epsilon^*$ and any $\rho > 0$ the controller (10) asymptotically stabilizes the equilibrium point $0$ with a domain of attraction containing $W$.

**Proof:** Consider any $(x(0)', \omega(0)')' \in W$. Then there exists a $T > 0$ independent of particular initial condition such that

$$\|C e_{(A+BF)}^t x(0)\| < \frac{1}{T}$$

for $t > T$. Denote

$$v(t) = -\delta \sigma(\frac{1+\rho}{\delta}) \left( B \right)' \left( P_\varepsilon \right) \left( x \right)$$

By construction, we have $\|v(t)\| \leq \delta$ for $t > 0$. This together with (5) implies that $\|z(t)\| \leq 1$ for $t > T$.

Since $A + BF$ is Hurwitz stable and the input to the second system is bounded, there exists a $W$ such that for any $(x(0)', \omega(0)')' \in W$, we have $(x(T)', \omega(T)')' \in W$.

Then the interconnection of (1) and (2) with controller (10) for $t > T$ is equivalent to the interconnection of (6) with controller (8) for $t > T$. From Lemma 1, there exists an $\epsilon^*$ such that for any $\epsilon \in (0, \epsilon^*)$ and any $\rho > 0$, the closed-loop system of (6) and controller (8) is asymptotically stable with $(x(T)', \omega(T)')' \in W$. Therefore we have

$$x(t) \rightarrow 0, \quad \omega(t) \rightarrow 0.$$ Since this follows for any $(x(0), \omega(0)) \in W$, we find that $W$ is contained in the domain of attraction as required. ■

**B. Global controller design**

We claim that the same controller given in (10) with $\epsilon$ being replaced by the scheduled low gain parameter $\varepsilon_s(\tilde{x})$ as defined below solves the global stabilization problem.

At first, we look for a scheduling parameter satisfying the following:

1) $\varepsilon_s(x) : \mathbb{R}^{n+m} \rightarrow (0, 1]$ is continuous and piecewise continuously differentiable.

2) There exists an open neighborhood $O$ of the origin such that $\varepsilon_s(x) = 1$ for all $x \in O$.

3) For any $\tilde{x} \in \mathbb{R}^{n+m}$, we have

$$\|\left( B \right)' \left( P_\varepsilon(\tilde{x}) \right) \tilde{x}\|_{\infty} \leq \delta.$$  

4) $\varepsilon_s(\tilde{x}) \rightarrow 0$ as $\|\tilde{x}\|_{\infty} \rightarrow \infty$.

5) $\{ \tilde{x} \in \mathbb{R}^{n+m} \mid \tilde{x}'P_\varepsilon(\tilde{x}) \tilde{x} \leq c \}$ is a bounded set for all $c > 0$.

6) $\varepsilon_s(\tilde{x})$ is uniquely determined given that $\tilde{x}'P_\varepsilon(\tilde{x}) \tilde{x} = c$ for some $c > 0$.

A particular choice satisfying the above criteria is given by

$$\varepsilon_s(\tilde{x}) = \max \{ \varepsilon \in (0, 1) \mid (\tilde{x}'P_\varepsilon(\tilde{x}) \tilde{x}) \text{ trace} \left[ \left( B \right)' \left( P_\varepsilon \right) \left( B \right) \right] \leq \delta^2 \}. $$

(11)

Then we first show the following result:

**Lemma 2** Consider the system (6) and assume that the pair $(A, B)$ as given by (3) is stabilizable and the eigenvalues of $M$ are in the closed left half plane. Then, for any $\rho > 0$, the feedback,

$$v = -\delta \sigma(\frac{1+\rho}{\delta}) \left( B \right)' \left( P_\varepsilon(\tilde{x}) \right) \tilde{x},$$

(12)

achieves global stability of the equilibrium point $\tilde{x} = 0$.  

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Proof: Obviously, controller (12) satisfies \( \|v\| < \delta \). It remains to show that such a controller achieves global stabilization. Define \( V(\tilde{x}) = \tilde{z}' P_{\varepsilon_s}(\tilde{x}) \tilde{x} \).

Denote
\[
\tilde{v} = \left( \begin{array}{c} B' \\ 0 \end{array} \right)' P_{\varepsilon_s}(\tilde{x}) \tilde{x}.
\]

Consider \( \tilde{V} \) along any trajectory,
\[
\dot{\tilde{V}} \leq -\tilde{x}' Q_{\varepsilon_s}(\tilde{x}) \tilde{x} - 2\tilde{\sigma} \tilde{v}' \left[ \sigma \left( \frac{(1+\rho)}{\delta} \tilde{v} \right) \right] + \tilde{x}' dP_{\varepsilon_s}(\tilde{x}) \tilde{x}.
\]

By construction, \( \|\tilde{v}\| < 1 \). We get
\[
\dot{\tilde{V}} \leq -\tilde{x}' Q_{\varepsilon_s}(\tilde{x}) \tilde{x} - 2\tilde{\sigma} \tilde{v}' \left[ \sigma \left( \frac{(1+\rho)}{\delta} \tilde{v} \right) \right] + \tilde{x}' dP_{\varepsilon_s}(\tilde{x}) \tilde{x}.
\]

If \( \rho > 0 \), we have
\[
\dot{V} < -\tilde{x}' Q_{\varepsilon_s}(\tilde{x}) \tilde{x} + \tilde{x}' dP_{\varepsilon_s}(\tilde{x}) \tilde{x}.
\]

The scheduling law (11) implies
\[
V(\bar{x}) \text{ trace } \left[ \left( \begin{array}{c} B' \\ 0 \end{array} \right)' P_{\varepsilon_s}(\bar{x}) \left( \begin{array}{c} B' \\ 0 \end{array} \right) \right] = \delta^2
\]
whenever \( \varepsilon_s(\bar{x}) \neq 1 \) or equivalently \( P_{\varepsilon_s}(\tilde{x}) \) is not a constant locally. This implies that \( \dot{V} \) and \( \tilde{x}' dP_{\varepsilon_s}(\tilde{x}) \tilde{x} \) are either both zero or of opposite signs. Hence for \( x \neq 0 \)
\[
\dot{V} < 0
\]
If not, we know \( \tilde{x}' dP_{\varepsilon_s}(\tilde{x}) \tilde{x} \leq 0 \). But this implies \( \dot{V} < -\tilde{x}' Q_{\varepsilon_s}(\tilde{x}) \tilde{x} \) which yields a contradiction. Therefore, the global asymptotic stability follows.

**Theorem 3** Consider the interconnection of the two systems given by (1) and (2) satisfying the conditions 1 and 2 of Theorem 1. Choose \( F \) such that \( A + BF \) is asymptotically stable. Let \( P_e \) and \( \varepsilon_s \) be as defined by (7) and (11) respectively. In that case, for any \( \rho > 0 \), the state feedback
\[
u = F x - \delta \sigma \left( \frac{(1+\rho)}{\delta} \tilde{v} \right) \tilde{x}
\]
achieves global asymptotic stability.

Proof: If we consider the interconnection of (1) and (2), then we note that close to the origin the saturation does not get activated. Moreover, close to the origin the feedback (13) is given by

\[
u = F x - (1 + \rho) \tilde{x}' P_{\varepsilon_s}(\tilde{x}) \tilde{x},
\]

which immediately yields that the interconnection of (1), (2) and (13) is locally asymptotically stable. It remains to show that we have global asymptotic stability.

Consider an arbitrary initial condition \( x(0) \) and \( \omega(0) \). Then there exists a \( T > 0 \) such that
\[
\|Ce^{(A+BF)t}x(0)\| < \frac{1}{2}
\]
for \( t > T \). Moreover, by construction, the control
\[
v = -\delta \sigma \left( \frac{(1+\rho)}{\delta} \tilde{v} \right) \tilde{x}
\]
yields \( \|v(t)\| \leq \delta \) for all \( t > 0 \). However, this implies that \( z(t) \) generated by (4) satisfies \( \|z(t)\| < 1 \) for all \( t > T \).

But this yields that the interconnection of (1) and (2) with controller (13) achieves local and high-gain feedback controller. Therefore, we expect an enhanced system performance from our design technique. A numerical example is given in Section IV to illustrate this result.

We like to emphasize that an appropriate selection of the matrix \( Q_\varepsilon \) plays an important role in the design process. A judicious choice of \( Q_\varepsilon \) can tremendously improve the performance. This is also illustrated by an example given in next section.

**IV. Example**

**A. Example 1: Semi-global stabilization via state feedback**

Consider the two systems \( L_1 \) and \( L_2 \) given in (1) and (2),

\[
L_1 : \begin{cases}
\dot{x}(t) = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \\
\gamma(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t),
\end{cases}
\]

and

\[
L_2 : \dot{\omega}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \omega(t) + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \sigma(\omega(t)).
\]

We design below a controller that stabilizes the cascaded system of \( L_1 \) and \( L_2 \) with an a priori given compact set \( \mathcal{W} \) to be contained in the domain of attraction of the closed-loop system, where \( \mathcal{W} = \{ \xi \in \mathbb{R}^4 \mid [-3,3]^4 \} \).

Step 1. Choose
\[
F = \begin{pmatrix} -22.2474 & -8.4495 \\ -22.2474 & -8.4495 \end{pmatrix}
\]
such that \( A + BF \) is Hurwitz stable.

Step 2. Choose \( \delta = 2.0772 \) and \( \rho = 1000 \). Then for system (4), we have
\[
\|v(t)\| < \delta \quad \forall t > 0,
\]
implying that \( \|z(t)\| < \frac{1}{2} \) for all \( t > 0 \).
Step 3. We set the low gain parameter $\varepsilon = 10^{-4}$. Choose $Q_\varepsilon = \varepsilon I$. After solving the associated algebraic Riccati equation, we obtain the following state feedback controller:

$$u = \begin{pmatrix} -22.2474 & -8.4495 \end{pmatrix} x - 1.0491 \sigma \{(386.5181 & 25.1874) x + (4.8190 & -198.2508) \omega \}.$$

For comparison purpose, a low gain feedback controller of the form,

$$u = F x + \begin{pmatrix} B' \end{pmatrix} P_\varepsilon (x \omega),$$

is also given as

$$u = \begin{pmatrix} -23.0495 & -8.5018 \end{pmatrix} x + \begin{pmatrix} 0.0100 & 0.4114 \end{pmatrix} \omega.$$

The simulation data is shown in Figure 3. For comparison, the simulation data of low-gain controller is shown in Figure 4. As we can see, the low-high gain enhances the performance by incurring much lower overshoot and undershoot.

\[\text{Fig. 3. Semi-global stabilization via state feedback-low high gain approach}\]

\[\text{Fig. 4. Semi-global stabilization via state feedback-low gain approach}\]

**B. Example 2: Global stabilization via state feedback**

The two systems $L_1$ and $L_2$ in (1) and (2) are the same as in the preceding example. We solve the global stabilization problem as follows:

Step 1. Choose

$$F = \begin{pmatrix} -22.2474 & -8.4495 \end{pmatrix}$$

such that $A + BF$ is Hurwitz stable.

Step 2. Choose the same $\delta = 2.0772$ as in the preceding example and $\rho = 1000$.

Step 3. Design a controller

$$u = F x - \delta \sigma \begin{pmatrix} 1+\rho \end{pmatrix} P_{\varepsilon}(\bar{x}) \bar{x}$$

where $P_{\varepsilon}(\bar{x})$ is given by (7) and (11).

The resulting simulation is shown in Figure 5. For comparison, the simulation data of a closed-loop system under a scheduled low gain feedback controller is shown in Figure 6. Clearly, the dynamics achieved by the low-and-high gain feedback has a lower overshoot.

\[\text{Fig. 5. Global stabilization via low-and-high state feedback}\]

\[\text{Fig. 6. Global stabilization via low gain state feedback}\]

**C. Example 3: The impact of $Q_\varepsilon$**

Consider the same system as used in Examples 1 and 2. Choose the same $F$ and henceforth we have the same $\delta$. 

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We observe that in above examples, the first state element of system $L_2$ has the worst performance. Therefore, instead of $Q_\varepsilon = \varepsilon I$, we use

$$Q_\varepsilon = \begin{pmatrix} \varepsilon I & 0 \\ 0 & 200\varepsilon I \end{pmatrix},$$

However, with this $Q_\varepsilon$, we have to choose a relatively smaller $\varepsilon$. Set $\varepsilon = 6 \times 10^{-6}$. After solving Algebraic Riccati equation, we obtain the following low-and-high gain feedback:

$$u = \begin{pmatrix} -12 & -6 & -7 \end{pmatrix} x - 2.28\sigma \left\{ \begin{pmatrix} 743.4252 & 46.9858 \\ -16.6934 & -391.1155 \end{pmatrix} x + \begin{pmatrix} 25 & 20 \end{pmatrix} \right\}$$

Then we re-examine the semi-global stabilization of the interconnection of $L_1$ and $L_2$ via low-and-high gain state-feedback and low gain state feedback respectively. The simulation data are shown in Fig 7 and Fig 8. This illustrates that, with a proper choice of $Q_\varepsilon$, we can refine the dynamics.

![Fig. 7. Semi-global stabilization via state feedback–low-and-high gain feedback with modified $Q_\varepsilon$](image1)

![Fig. 8. Semi-global stabilization via state feedback–low gain feedback with modified $Q_\varepsilon$](image2)

V. CONCLUSIONS

The problems of semi-global and global internal stabilization of a class of sandwich systems consisting of two linear systems with a saturation element in between are revisited. Low-and-high gain and scheduled low-and-high gain state feedback design methodologies to solve these stabilization problems are developed. Such design methods can be used successfully for control problems beyond stabilization, to enhance performance such as robust stability and disturbance rejection. Numerical examples are presented to illustrate the results.

REFERENCES