QUEUEING NETWORKS WITH STRING TRANSITIONS OF MIXED VECTOR ADDITIONS AND VECTOR REMOVALS

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Abstract. Product form queueing networks with string transitions have been studied in the literature as a model incorporating several features of the existing networks. That model includes state-dependent transition rates at the cost of a restrictive form of the string transitions. First a sequence of nonnegative vectors is removed, and then a sequence of nonnegative vectors is added to the network state. Such a transition structure excludes, for example, networks with positive and negative signals recently studied in the literature. This paper extends the string transition networks to allow transitions of mixed vector additions and vector removals, and it includes assembly-transfer networks as well as networks with negative and positive signals as special cases. Assuming that the transition rates are independent of the network state except at the boundaries, we obtain general modifications for the string transition network under which it possesses a product form equilibrium distribution. The network is shown to satisfy a class of local balance as expressed by a set of traffic equations.

Key words. Queueing networks, boundary modifications, product form solutions, string transitions.

1 Introduction

Product form queueing networks have regained considerable interest over the last couple of years. In particular, extensions beyond the well-known Jackson, BCMP, and Kelly-Whittle type networks (e.g., [1–5]) have been established. Extensions include networks with batch movements (such as [6–10]), networks with negative customers and signalling ([11–17], among others), and assembly-transfer networks ([18, 19]). Transitions in product form networks now generally include multiple stages. A framework based on string transitions is provided in Serfozo and Yang [20] and extended in Chao [21]. Their Markov network processes provide a description of networks and include a number of models studied in the literature. One major limitation of the models of [20] and [21] is that the type of transitions must take the following form. First, a sequence of nonnegative vectors is subtracted from the network state, and then a sequence of nonnegative vectors is added. This restriction excludes many network models with multiple

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batch additions, such as the network with positive and negative signals studied by Chao and Miyazawa [13].

The present paper considers a network with string transition of mixed vector additions and vector removals. Unlike the network of [20], such networks, even in the simplest form of single node case, generally do not have a product form equilibrium distribution. Extra conditions for the network to possess a tractable solution, therefore, have to be introduced (which is in spirit similar to the networks in Miyazawa and Taylor [19] and Chao [21]). Assuming that the transition rates are independent of the network state except at the boundaries, we obtain general boundary modifications for the string transition network under which it possesses a product form equilibrium distribution.

The first instance that analyzes an analytically intractable network by introducing extra arrivals is due to Miyazawa and Taylor [19]. In [19] an assembly-transfer network with batch arrival and batch service is considered. It is shown that if an additional arrival process is turned on whenever a node is empty, the network has a simple geometric product form solution. In an earlier paper [21] we extend the framework of Serfozo and Yang [20] by considering a network with string transitions that begins with a sequence of vector removals followed by a sequence of vector additions. The present paper is motivated by, and extends the results of [19] and [21]. The string transitions considered in this paper take an arbitrary form and it has mixed vector removals and vector additions. We consider a model in which a string is interrupted when the effect of a transition vector results in a state outside of the state space. Thus a partial transition occurs and it takes the process to the last state before the string leaves the state space. To compensate for the change in probability flow due to partial strings, extra strings are generated in states at the boundary of the state space. Such strings play a role similar to the additional arrival rate in empty queues proposed in [10] and [21]. The modification is possible as a result of the traffic equations expressing partial balance properties of string transition networks.

Product form queuing networks have been used as approximate models for evaluating performance of networks and obtaining qualitative insight into network behavior. As a first step, such approximations usually include modifications of transition rates near the boundaries of the state space. A second step might include bounds indicating the accuracy of the resulting approximation. The present paper provides a result for the first step in this approximation procedure. Bounds for assembly-transfer networks have been reported in [19], but a unified framework for obtaining bounds for string transition networks is not yet available. Our results not only characterize a new class of models with a product form equilibrium distribution, but also provide a generic construction of product form models based on partial balances that are useful in developing approximations.

In Section 2 we present the model with compensating boundary rates, and the results are illustrated in Section 3 via networks with signals, and assembly-transfer networks. We also discuss reflecting boundaries as an alternative model in Section 3.

2 The Main Result

Consider a homogeneous, irreducible, positive recurrent Markov chain \( X = (X(t), t \geq 0) \) with state space \( \mathcal{E} \) that records the (possibly negative) number of units at the nodes of a stochastic network, where \( \mathcal{E} \) is an arbitrary subset of \( \mathbb{Z}^N \), that is not even required to be convex. A state is an \( N \)-dimensional vector \( \mathbf{z} = (z_1, z_2, \ldots, z_N) \), where \( z_i \) denotes the number of units at node \( i \). Let \( q(x, y) \), \( x, y \in \mathcal{E} \), denote the transition rate from state \( x \) to state \( y \).

Transitions of \( X \) are determined by a set of strings \( \mathcal{S} \). Each string \( s \in \mathcal{S} \) takes the form

\[
\mathcal{E} = \mathcal{E}_1^{(s_1)} \mathcal{E}_2^{(s_2)} \cdots \mathcal{E}_n^{(s_n)},
\]
where $s_i \in \mathcal{A}$ and $\mathcal{A}$ is a set of nonnegative vectors in $\mathbb{Z}_+^N = \{0, 1, 2, \cdots \}^N$, $e_i \in \{-1, 1\}$ indicates whether the increment vector $s_i$ is added ($e_i = -1$), or deleted ($e_i = 1$), and $\ell(s)$ is the length of string $s$. It is assumed that $\mathcal{S}$ contains the zero string, 0, of length zero. When the state of the process is $x$, an $s$ string transition occurs at rate $\lambda_s$. For convenience, $\lambda_s$ is understood as zero whenever $s \notin \mathcal{S}$. For $s \in \mathcal{S}$, a full $s$ string transition of $X$ is defined by

$$x \to x - (e_1 s_1 + \cdots + e_{\ell(s)} s_{\ell(s)}).$$

The full string transition occurs when each 'intermediate state' $x - (e_1 s_1 + \cdots + e_k s_k), 1 \leq k \leq \ell(s)$, is in the state space $\mathcal{E}$. Alternatively, if at an intermediate stage of the string transition the process jumps out of the state space $\mathcal{E}$, then the string transition is interrupted, resulting in a partial string transition

$$x \to x - (e_1 s_1 + \cdots + e_k s_k), 1 \leq k < \ell(s),$$

which occurs if

$$x - (e_1 s_1 + \cdots + e_k s_k) \in \mathcal{E}, \quad i = 1, 2, \cdots, k, \quad \text{and} \quad x - (e_1 s_1 + \cdots + e_k s_{k+1}) \not\in \mathcal{E},$$

i.e., the string ends at the last point before it jumps out of the state space.

For convenience, define $ss'$ as the concatenation of $s$ and $s'$. That is, for strings $s = s_1 s_2 \cdots s_{\ell(s)} \in \mathcal{S}$ and $s' = s'_1 s'_2 \cdots s'_{\ell(s')}$,

$$ss' \equiv s_1 s_2 \cdots s_{\ell(s)} s'_1 s'_2 \cdots s'_{\ell(s')}.$$

Furthermore, to simplify notation, denote

$$x - \bar{s}(i) = x - (e_1 s_1 + \cdots + e_i s_i), \quad i = 1, \cdots, \ell(s),$$

and, for $i = \ell(s)$ simply write

$$x - s \equiv x - s(\ell(s)).$$

This notation will also be used for partial strings (when it does not lead to confusion), i.e., for string $\bar{s} = ss'$, denote $x - s = x - s(\ell(s))$.

The transition rate of the Markov chain $X$ due to string $s$ is, for $x \in \mathcal{E}$,

$$r_s(x, x - s) = \lambda_s [x - s(i) \in \mathcal{E}, \quad i = 1, 2, \cdots, \ell(s)],$$

where $1[\text{event}]$ is the indicator of 'event'. For a partial string the transition rate is, for $x \in \mathcal{E}$,

$$r_s(x, x - s(k)) = \lambda_s [x - s(i) \in \mathcal{E}, \quad i = 1, 2, \cdots, k, \quad x - s(k+1) \not\in \mathcal{E}].$$

We remark that when $e_i = 1, i = 1, 2, \cdots, \ell(s) - 1$, and $e_{\ell(s)} = -1$, i.e., string $s$ first removes $\ell(s) - 1$ vectors which is then followed by the addition of one vector, the model reduces to that of [20]. On the other hand, if $e_i = 1$ for $i = 1, 2, \cdots, \ell(s) - u$, and $e_i = -1$ for $i = \ell(s) - u + 1, \cdots, \ell(s)$, where $u$ is some number between 1 and $\ell(s)$, i.e., the string starts with the removal of a sequence of vectors followed by the addition of a sequence of vectors, then we obtain the model of [21]. In particular, if $\mathcal{A} = \{e_i, i = 1, 2, \cdots, N\}$, and $s \in \mathcal{S}$ takes the form $s = (e_1 e_2 \cdots e_i)(e_{i-1} e_{i-2} \cdots e_1)$, then we obtain the assembly-transfer networks of Miyazawa and Taylor [19]. Such processes are known to not have a product form solution. Thus, following [19], we introduce extra conditions to make the model mathematically tractable. Interaction with the boundary occurs due to partial strings terminating at states $x \in \mathcal{E}$ such that $x - ea \notin \mathcal{E}$.
for some $a \in A$, $c \in \{-1,1\}$. Therefore, we introduce additional boundary rates compensating for partial strings. To this end, let

\begin{align}
  r_s(x, x + s(i) - s(k)) &= \beta_s(i)1[x + c_is_i \notin E, x + s(i) - s(j) \notin E, \\
  & \quad \text{for } j = i + 1, \ldots, k, \text{ and } x + s(i) - s(k + 1) \notin E], \\
  r_s(x, x + s(i) - s(\ell(s))) &= \beta_s(i)1[x + c_is_i \notin E, x + s(i) - s(j) \notin E, \\
  & \quad \text{for } j = i + 1, \ldots, \ell(s)].
\end{align}

(1)

(2)

Similar boundary rates were introduced in Miyazawa and Taylor [19], also see Chao [18], for assembly-transfer networks, and in Chao [21]. As illustrated in Figure 1, compensating partial strings are such that a string that enters the state space in state $x$, but that, in the unconstrained case, would have originated in state $x + s(i)$ outside the state space, is now considered as a compensating string originating in $x$ and ending in $x + s(i) - s(\ell(s))$. Alternatively, a compensating string might also leave the state space. Then, let $x + s(i) - s(k)$ be the last state the string visits before leaving the state space; a compensating string transition rate is added from $x$ to $x + s(i) - s(k)$.

![Diagram](image)

Figure 1. Extra rates for boundary state $x$

Assume $\sum_{s \in S} \lambda_s < \infty$. Then the Markov chain is regular, implying that it is uniquely determined by its transition rates. The transition rate from $x \in E$ to $y \in E$ is a combination of the just defined rates, and is given by

$$
\psi(x, y) = \sum_{s \in S} r_s(x, y).
$$

The following result is frequently used in the proof of Theorem 1, and is therefore stated separately. Its proof is elementary and is omitted.

**Lemma 1** The following identity is satisfied for any sequence $x(1), x(2), \ldots, x(n)$:

1. $1[x(i) \in E, i = 1, 2, \ldots, n] + \sum_{j=1}^{n} 1[x(\ell) \in E, \ell < j, x(j) \notin E] = 1,$
2. $1[x(i) \in E, i = 1, 2, \ldots, n] + \sum_{j=1}^{n} 1[x(\ell) \in E, \ell > j, x(j) \notin E] = 1.$

The following is the main result of this section.
Theorem 1 Let $\eta : E \rightarrow \mathbb{R}_+$ be a solution of the traffic equations:

$$\eta(a') \sum_{s', s''} \eta(s) \lambda_{s''} = \sum_{s' s''} \eta(s) \lambda_{s''}, \quad \text{for all } a \in A \setminus \{0\}, \quad c \in \{-1, 1\},$$

(3)

where $\eta(s) \equiv \eta(s(\ell(s)))$. If $\eta$ is exponential, i.e., $\eta(x + y) = \eta(x)\eta(y)$ for all $x, y \in E$, and

$$\lambda_s \eta((s(i)) - \beta_s(i), \quad i = 1, 2, \ldots, \ell(s), \quad s \in S,$

(4)

then $\pi(x) = B \eta(x), x \in E$, is the unique equilibrium distribution of $X$, where $B$ is a normalizing constant.

Proof It suffices to show that $\pi$ satisfies the global balance equations, for $x \in E$,

$$\pi(x) \sum_{y \in E} q(x, y) = \sum_{y \in E} \pi(y)q(y, x).$$

For $x$ in the interior of $E$, i.e., $x$ is such that $x - \epsilon a \in E, \epsilon \in \{-1, 1\}$, for all $a \in A$, the process can go out of $x$ due to a complete or partial string transition. Thus we have

$$L H S(x) = \pi(x) \sum_{y \in E} q(x, y)$$

$$= \pi(x) \sum_y \sum_k r_y(x, y) \mathbf{1}[y = x - s(k)]$$

$$= \pi(x) \sum_s \left\{ \sum_{k=1}^{\ell(s)-1} \lambda_s \mathbf{1}[x - s(i) \in E, i = 1, 2, \ldots, k, x - s(k+1) \notin E] + \lambda_s \mathbf{1}[x - s(i) \in E, i = 1, 2, \ldots, \ell(s)] \right\}$$

(5)

where the last equality follows from Lemma 1. For the right hand side of the global balance equations, for an interior point $x$ there are two types of transitions that bring the process to state $x$ (see Figure 2):

![Figure 2](image-url)
(i) a transition from \( x + s \) due to string \( s \) that stays in \( \mathcal{E} \) during the string transition, and thus ends in \( z \);

(ii) a transition into \( x \) due to modified rate (2); this string starts in a state \( x + s - s(i) \) such that \( x + s - s(i - 1) \notin \mathcal{E} \), and ends at \( x \).

Therefore, the rate into \( x \) is

\[
RHS(x) = \sum_{y \in \mathcal{E}} \tau(y) q(y, x)
\]

\[
= \sum_{y} \sum_{s \in \mathcal{S}} \sum_{k=0}^{\ell(s)-1} n(y) r_s(y, x) 1[y = x + s - s(k)]
\]

\[
= \sum_{s} \left\{ \sum_{k=1}^{\ell(s)-1} \tau(x + s - s(k)) \beta_s(k) 1[x + s - s(k - 1) \notin \mathcal{E}] \right. \\
\times 1[x + s - s(i) \in \mathcal{E}, i = k, \ldots, \ell(s)] \\
+ \tau(x + s) \lambda_s 1[x + s - s(i) \in \mathcal{E}, i = 0, \ldots, \ell(s)] \right\}
\]

\[
= \tau(x) \sum_{s} \lambda_s \eta(s) \left\{ \sum_{k=1}^{\ell(s)-1} 1[x + s - s(k - 1) \notin \mathcal{E}] \right. \\
\times 1[x + s - s(i) \in \mathcal{E}, i = k, \ldots, \ell(s)] \\
+ 1[x + s - s(i) \in \mathcal{E}, i = 0, \ldots, \ell(s)] \right\}
\]

\[
= \tau(x) \sum_{s} \lambda_s \eta(s), \tag{6}
\]

where the last equality follows from Lemma 1, and because \( x - s_1 s_1 \in \mathcal{E} \), \( x + \epsilon_1 s_1 s_k(s) \in \mathcal{E} \), which follows from the assumption that \( x \) is an interior point of \( \mathcal{E} \). By (5) and (6), for an interior point \( x \), it suffices to show that

\[
\sum_{s} \lambda_s = \sum_{s} \lambda_s \eta(s). \tag{7}
\]

To this end, observe that

\[
\sum_{s'} \lambda_{s'} = \lambda_s + \sum_{s \neq s'} \sum_{a \neq 0} \lambda_{sa + 1 s'} + \sum_{a \neq 0} \lambda_{sa - 1 s'}, \tag{8}
\]

and

\[
\sum_{s,s'} \eta(s) \lambda_{ss'} = \sum_{s'} \lambda_{s'} + \sum_{s,s'} \sum_{a \neq 0} \eta(s) \eta(a^{-1}) \lambda_{sa + 1 s'} + \sum_{s,s'} \eta(s) \eta(a^{-1}) \lambda_{sa - 1 s'}. \tag{9}
\]

Substituting (8) into the left-hand side of (9) and rearranging terms gives

\[
\sum_{s} \eta(s) \lambda_s - \sum_{s} \lambda_s = \sum_{s \neq 0} \left[ \sum_{s,s'} \eta(s) \eta(a^{-1}) \lambda_{sa + 1 s'} - \sum_{s,s'} \eta(s) \lambda_{sa - 1 s'} \right] (1 - \eta(a^{-1})).
\]
Thus, by the traffic equation (3), the left hand side of this equation has to be zero, which establishes (7).

For states on the boundary of $\mathcal{E}$, i.e., states $x$ such that there exists $a$ for which $x - \epsilon_a \notin \mathcal{E}$ for $c = 1$ or $c = -1$, the rates into and out of a state $x$ contain more terms than in the interior point case. We first calculate the rate out of state $x$. For each $s \in \mathcal{S}$ such that $x - \epsilon_1 s_1 \in \mathcal{E}$, the process has rate $\lambda_s$ out of $x$, and therefore $LHS(x)$ contains, recalling (5),

$$\pi(x) \sum_s \lambda_s 1[x - \epsilon_1 s_1 \in \mathcal{E}]. \quad (10)$$

In addition, due to the additional compensating stringe, there are extra rates out of $x$ (see Figure 1):

(i) due to string $s$ entering $\mathcal{E}$ at $x$, and staying in $\mathcal{E}$ afterwards, i.e., $x + \epsilon_1 s_1 \notin \mathcal{E}$, but $x + s(i) - s(j) \in \mathcal{E}$ for $j = i + 1, \ldots, \ell(s)$;

(ii) due to string $s$ entering $\mathcal{E}$ at $x$, and leaving $\mathcal{E}$ afterwards, i.e., $x + \epsilon_1 s_1 \notin \mathcal{E}$, $x + s(i) - s(j) \in \mathcal{E}$, $j = i + 1, \ldots, k$ for some $k < \ell(s)$, but $x + s(i) - s(k + 1) \notin \mathcal{E}$.

Thus the rate out of $x$ due to the modified arrivals is

$$\pi(x) \sum_s \sum_{i=1}^{\ell(s)-1} \beta_s(i) 1[x + \epsilon_1 s_1 \notin \mathcal{E}] \left\{ \sum_{h=i+1}^{\ell(s)-1} 1[x + s(i) - s(j) \in \mathcal{E}, j = i + 1, \ldots, k, x + s(i) - s(k + 1) \notin \mathcal{E}] + 1[x + s(i) - s(j) \in \mathcal{E}, j = i + 1, \ldots, \ell(s)] \right\}$$

$$= \pi(x) \sum_s \lambda_s \sum_{i=1}^{\ell(s)-1} \eta(s(i)) 1[x + \epsilon_1 s_1 \notin \mathcal{E}] 1[x - \epsilon_{i+1} s_{i+1} \in \mathcal{E}], \quad (11)$$

where the last equation follows, again, from Lemma 1. Combining (10) and (11), the rate out of state $x$ is:

$$\pi(x) \left( \sum_s \lambda_s 1[x - \epsilon_1 s_1 \in \mathcal{E}] + \sum_s \lambda_s \sum_{i=1}^{\ell(s)-1} \eta(s(i)) 1[x + \epsilon_1 s_1 \notin \mathcal{E}] 1[x - \epsilon_{i+1} s_{i+1} \in \mathcal{E}] \right). \quad (12)$$

For boundary state $x$, the rate into state $x$ contains the following terms (see Figure 3):

(i) string transitions from $x + s$ to $x$ that stay in $\mathcal{E}$, with corresponding rate $\pi(x + s) \lambda_s$;

(ii) compensating string transitions from $x + s - s(i) \in \mathcal{E}$ (for some $1 < i < \ell(s)$) to $x$ for which $x + s - s(i - 1) \notin \mathcal{E}$ and $x + s - s(k) \in \mathcal{E}$ for $i < k \leq \ell(s)$, with rate $\pi(x + s - s(i)) \beta_s(i)$;

(iii) partial string transitions from $x + s(i)$ to $x$ (for some $1 \leq i < \ell(s)$) terminating because $x - \epsilon_{i+1} s_{i+1} \notin \mathcal{E}$, with rate $\pi(x + s(i)) \lambda_s$;

(iv) compensating string transitions that leave $\mathcal{E}$ at $x$. Such transitions originate at $x + s(i) - s(k)$ due to a modified rate because $x + s(i) - s(k - 1) \notin \mathcal{E}, x - \epsilon_{i+1} s_{i+1} \notin \mathcal{E}$, and $x - s(i) + s(u) \in \mathcal{E}, k \leq u \leq i$. The corresponding rate is $\pi(x + s(i) - s(k)) \beta_s(k)$. 


The rates due to (i) and (ii) concern string transitions terminating at \( x \) for which \( x + \epsilon_{\ell(s)} s_{\ell(s)} \) is in \( \mathcal{E} \). Combining these rates and applying Lemma 1 gives the following contribution to \( RHS(x) \), also recalling (9):

\[
\tau(x) \sum_s \lambda_s \eta(s) \mathbf{1}[x + \epsilon_{\ell(s)} s_{\ell(s)} \in \mathcal{E}].
\] (13)

The rates due to (iii) and (iv) can be written as

\[
\sum_s \sum_{i=1}^{\ell(s)-1} \pi(x + s(i)) \lambda_s \mathbf{1}[x + s(i) - s(j) \in \mathcal{E}, j = 0, \cdots, i, x - \epsilon_{i+1} s_{i+1} \notin \mathcal{E}]
\]

\[
+ \sum_s \sum_{i=1}^{\ell(s)-1} \sum_{k=1}^{i-1} \pi(x + s(i) - s(k)) \beta_s(k)
\]

\[
\times \mathbf{1}[x + s(i) - s(j) \in \mathcal{E}, j = k, \cdots, i, x + s(i) - s(k-1) \notin \mathcal{E}, x - \epsilon_{i+1} s_{i+1} \notin \mathcal{E}]
\]

\[
= \pi(x) \sum_s \lambda_s \sum_{i=1}^{\ell(s)-1} \eta(s(i)) \mathbf{1}[x + s(i) - s(j) \in \mathcal{E}, j = 0, \cdots, i, x - \epsilon_{i+1} s_{i+1} \notin \mathcal{E}]
\]

\[
+ \pi(x) \sum_s \lambda_s \sum_{i=1}^{\ell(s)-1} \eta(s(i))
\]

\[
\times \mathbf{1}[x + s(i) - s(j) \in \mathcal{E}, j = k, \cdots, i, x + s(i) - s(k-1) \notin \mathcal{E}, x - \epsilon_{i+1} s_{i+1} \notin \mathcal{E}]
\]

\[
= \pi(x) \sum_s \lambda_s \eta(s(i)) \mathbf{1}[x + \epsilon_s s_i \in \mathcal{E}] \mathbf{1}[x - \epsilon_{i+1} s_{i+1} \notin \mathcal{E}],
\] (14)

where the last equality follows from Lemma 1.

Combining (13) and (14) yields the rate into a boundary state \( x \). Thus, from these equations
and (12) the global balance equations read, for a boundary state $x$:

$$
\pi(x) \left( \sum_s \lambda_s \mathbf{1}[x - c_s s \in \mathcal{E}] + \sum_s \lambda_s \sum_{i=1}^{\ell(s)-1} \eta(s(i)) \mathbf{1}[x + e_i s_i \notin \mathcal{E}] \mathbf{1}[x - e_{i+1} s_{i+1} \in \mathcal{E}] \right) = \pi(x) \sum_s \lambda_s \eta(s) \mathbf{1}[x + e_s s \in \mathcal{E}] + \pi(x) \sum_s \lambda_s \sum_{i=1}^{\ell(s)-1} \eta(s(i)) \mathbf{1}[x + e_i s_i \in \mathcal{E}] \mathbf{1}[x - e_{i+1} s_{i+1} \notin \mathcal{E}].
$$

(15)

Adding

$$
\sum_s \sum_{i=1}^{\ell(s)-1} \pi(x) \lambda_s \eta(s(i)) \mathbf{1}[x - e_{i+1} s_{i+1} \in \mathcal{E}] + x + e_i s_i \in \mathcal{E}]
$$

to both sides of this equation yields, for $x \in \mathcal{E}$,

$$
\pi(x) \sum_s \lambda_s \sum_{i=1}^{\ell(s)} \eta(s(i) + 1) \mathbf{1}[x - e_i s_i \in \mathcal{E}] = \pi(x) \sum_s \lambda_s \sum_{i=1}^{\ell(s)} \eta(s(i)) \mathbf{1}[x + e_i s_i \in \mathcal{E}] + x + e_i s_i \in \mathcal{E}]
$$

Now, relabel the string $s$ above by writing $a = s_i$ and $e = e_i$, the initial segment of $s$ before $s_i$ as $a$, and the tail segment after $s_i$ as $s'$. Then $x$ can be written as $sa - s'$ on the right hand side and as $sa - s'$ on the right hand side. Thus, the equation can be rewritten as

$$
\sum_{a \in \mathcal{E}} \left[ \sum_{s,s'} \lambda_{sa - s'} \eta(s) \right] \mathbf{1}[x, x - ca \in \mathcal{E}] = \sum_{a \in \mathcal{E}} \left[ \sum_{s,s'} \lambda_{sa - s'} \eta(s) \eta(a^{-1}) \right] \mathbf{1}[x, x - ca \in \mathcal{E}].
$$

For $x \in \mathcal{E}$, $a \in \{-1, 1\}$, and $a \in \mathcal{A}$ fixed, this equation is satisfied as a consequence of the traffic equations (3). This completes the proof.

The traffic equations (3) can be interpreted as partial balance equations, equating for each vector $a \in \mathcal{A}$ the probability flow for strings releasing $a$, and strings depositing $a$:

$$
\sum_{s,s'} \eta(x + s + ca) \lambda_{sa - s'} = \sum_{s,s'} \eta(x + s) \lambda_{sa - s'},
$$

where we have extended the functional relation for $\eta$ to $\mathbb{Z}^N$. For $x \in \mathbb{Z}^N$ such that $x + ca \in \mathcal{E}$ these equations can be interpreted as balance of probability flow through the boundary due to increment vector $a$. It is this relation that forms the explanation of the result of Theorem 1. In addition, balance of probability flow through the boundary as obtained from the traffic equations can also be interpreted via reflecting boundaries, where a string leaving $\mathcal{E}$ due to increment vector $a$ is bounced back into the state space via another string that also contains increment vector $a$. This point is further discussed in the next section.

In the framework of Serfozo and Yang [20] the transition rates also contain a state-dependent triggering function $\phi_s(k)$ as introduced in Henderson [22] for networks with negative customers:

$$
r_s(x, y) = \lambda_s \phi_s((a))(x) \mathbf{1}(y = x - s) + \sum_{k=0}^{\ell(s)-1} \lambda_s \left[ \phi_s(k)(y - s) - \phi_s(k+1)(y) \right] \mathbf{1}(y = x - s(k)).
$$
For a proper definition, these rates require that \( \phi_s(x) \geq \phi_{s(k+1)}(x) \), for all \( k = 1, 2, \ldots, \ell(s) \), \( s \in S \), and \( x \in \mathcal{E} \). For string transitions terminating upon departure from the state space \( \mathcal{E} = \mathbb{Z}^N_+ \), it must be that \( \phi_s(x) \geq \phi_s(x - \epsilon a) \). For \( \epsilon = 1 \) this observation implies that \( \phi_s \) must be monotonically increasing in all components, whereas for \( \epsilon = -1 \) we obtain that \( \phi_s \) must be monotonically decreasing. Hence, at the state space \( \mathcal{E} = \mathbb{Z}^N_+ \), \( \phi_s \) must be a constant, which we have set equal to one. Thus, in the framework of Serfozo and Yang [20] the increment vectors must be positive to allow for state-dependent transition rates. The same comment applies to the model of Chao [21]. The model of this paper does not contain a state-dependent triggering function, but it allows for positive and negative increment vectors.

Now consider the network of Serfozo and Yang [20] restricted to state-independent transition rates modeling state space constraints only, i.e., \( \phi_s(x) = 1(x = s(k) \in \mathcal{E}) \). For this network string \( s \in S \) induces a transition \( s \rightarrow s - (s_1 + \cdots + s_{\ell(s)}) + a \). As a consequence, the traffic equations read (in our notation), for \( a \in A \),

\[
\eta(a^+) \sum_{s', s} \eta(s) \lambda_{s_{\ell(s)}+1} = \sum_{s \in S} \eta(s) \lambda_{s_{\ell(s)} - 1}
\]

because a string cannot have a negative increment vector other than the last vector in the sequence, i.e., \( \lambda_{s_{\ell(s)} - 1} \equiv \lambda_{s_{\ell(s)} - 1} \). Further observe that the additional boundary rate \( \beta_{s}(i) \) is defined for those strings that 'enter' the state space due to the \( i \)-th increment vector \( s_i \). Such strings cannot occur in the network of Serfozo and Yang [20]. Thus no extra arrivals are needed in [20]. Similarly, if we apply Theorem 1 to the model of Chao and Serfozo [21] with \( \Psi = \Phi \equiv 1 \), then we obtain the result of [21] which requires additional arrivals of batch \( a \) at the boundary.

3 Examples and Discussions

The model presented in this paper includes many known results as special cases. For instance, it includes networks with customers and signals, as long as the transition rates are not state-dependent. Similarly, our model includes all the examples presented in Serfozo and Yang [20]. In this section we illustrate our results by two instances that are not included in the framework of [20]. We also discuss an alternative boundary modification, called boundary reflection.

3.1 Assembly-transfer networks

Assembly-transfer networks with product form equilibrium distribution were recently introduced in Miyazawa and Taylor [19], and partial balance for such networks was derived in Chao [18]. Both the product form result and the partial balance property are due to an additional stream of arriving batches generated at an empty queue. This example justifies that the additional arrival stream of Miyazawa and Taylor [19] is of the form introduced in Section 2.

Consider a queueing network with \( N \) nodes. Batches of customers arrive at node \( j \) according to a Poisson process with rate \( \nu_j \), \( j = 1, 2, \ldots, N \). Batch sizes are i.i.d. random variables with distribution \( P(A_j = n) = c_j(n) \), \( n = 1, 2, \ldots \), where \( A_j \) is the generic batch size at node \( j \), and \( \sum_{n=1}^{\infty} c_j(n) = 1 \). The service times at node \( j \) are exponentially distributed with rate \( \mu_j \), \( j = 1, 2, \ldots, N \). At each node, customers are served in batches. Batch sizes are i.i.d. random variables with distribution \( P(B_j = n) = b_j(n) \), \( n = 1, 2, \ldots \), where \( B_j \) is the generic batch size at node \( j \), and \( \sum_{n=1}^{\infty} b_j(n) = 1 \). Upon service completion at node \( j \), if \( B_j = k \) and the number of customers present at node \( j \) is at least \( k \), the number of customers at node \( j \) is reduced by \( k \); these customers will be routed to node \( i \) and transferred into a batch of \( m \) customers with probability \( p_{j,k}(i,m) \), and these customers will leave the network with probability \( p_{j,k,m} \), where \( \sum_{i=1}^{N} \sum_{m=1}^{\infty} p_{j,k}(i,m) + p_{j,k,m} = 1 \), for \( j = 1, 2, \ldots, N \), and \( k = 1, 2, \ldots \). Otherwise, if
the number of requested customers present at node \( j \) is less than the requested \( k \) customers, the node is emptied, and all customers at node \( j \) are deleted from the network.

Assume that whenever node \( j \) is empty, an additional arrival process is turned on. This additional arrival process follows a batch Poisson process with rate \( \nu_j^* \). The batch size is a random variable with distribution \( P(A_j^* = n) = c_j^*(n), \ n = 1, 2, \ldots, \) where \( A_j^* \) is the generic batch size at node \( j \), and \( \sum_{j=1}^{\infty} c_j^*(n) = 1 \). Assume that equations

\[
\begin{align*}
\nu_j \left( \sum_{n=1}^{\infty} c_j(n) \rho_j^n - 1 \right) + \mu_j \left( \sum_{n=1}^{\infty} b_j(n) \rho_j^n - 1 \right) \\
= \sum_{i=1}^{N} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \mu_i b_i(k) p_{i,k}(j,m) \rho_i^k (1 - \rho_j^{-m}) \quad j = 1, 2, \ldots, N, \tag{16}
\end{align*}
\]

have a solution \( \rho_j \in (0,1), j = 1, 2, \ldots, N \). Under the condition that the additional arrival process is given by

\[
\begin{align*}
\nu_j^* &= \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \alpha_j(m) \rho_j^{n-m}, \\
c_j^*(n) &= \frac{\sum_{m=n+1}^{\infty} \alpha_j(m) \rho_j^{n-m}}{\sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \alpha_j(m) \rho_j^{n-m}}
\end{align*}
\]

where \( \alpha_j(n) \) is the average arrival rate of batches of size \( n \) at node \( j \), i.e.,

\[
\alpha_j(n) = \nu_j c_j(n) + \sum_{i=1}^{N} \sum_{k=1}^{\infty} \mu_i b_i(k) p_{i,k}(j,n),
\]

and the regularity assumption that the probability generating functions for \( c_j(n) \) and \( p_{i,k}(j,n) \) have radius of convergence at least 1, Miyazawa and Taylor [19] show that the network has the geometric product form distribution

\[
\pi(x) = \prod_{k=1}^{N} (1 - \rho_j) \rho_j^x, \quad x \in \mathbb{Z}_+^N.
\]

This result can also be concluded from the model of Section 2. To this end, let \( \mathcal{A} = \{e_i, \ i = 1, 2, \ldots, N\} \), where \( e_i \) is the \( i \)-th unit vector, with a 1 in place \( i \), zero elsewhere. A string \( x \in \mathcal{S} \) then either has the form \( s = a^1 a^2 \cdots a^i, \ a \in \mathcal{A}, \) or \( s = a^{-1} \cdots a^{-1}(a')^{-1} \cdots (a')^{-1}, \ a, a' \in \mathcal{A}, \) with rates

\[
\lambda_x = \begin{cases} 
\nu_i e_i(n) & s = (e_i)^{-1} \cdots (e_i)^{-1} \quad (n \text{ terms}), \\
\mu_i e_i(k) p_{i,k}(j,n) & s = (e_i)^{-1} \cdots (e_i)^{-1}(e_j)^{-1} \cdots (e_j)^{-1} \quad (k \text{ resp. } m \text{ terms}).
\end{cases}
\]

The traffic equations (3) then read, for \( a = e_j, \ \epsilon = -1, \)

\[
\eta'(e_j) \sum_{n=1}^{\infty} \sum_{k=0}^{m-1} \eta'(k e_j) \nu_j c_j(n) + \eta'(e_j) \sum_{i=1}^{N} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=0}^{m-1} \eta(n, e_i) \eta'(e_j) \mu_i b_i(n) p_{i,j}(j,m)
\]

\[
= \sum_{n=1}^{\infty} \sum_{k=0}^{m-1} \eta(k e_j) \nu_j b_j(n) \left\{ \sum_{i=1}^{N} \sum_{m=1}^{\infty} p_{i,j}(i,m) + p_{j,n} \right\},
\]
observing that the increment vector \( a = c_1 \) is contained in that part of a string in which node \( i \) receives customers (at position \( k + 1 \)). For \( \epsilon = 1 \) we obtain the same equations. Rearranging terms, with

\[
\eta(x) = \prod_{i=1}^{n} \beta_i^2, 
\]

and performing summation over \( k \) then reproduce equation (16), i.e., for this example our traffic equations (3) coincide with those of Miyazawa and Taylor [19].

For obtaining a product form equilibrium distribution, Theorem 1 requires the additional boundary rates \( \beta_j(i) = \lambda_j \eta(s(i)) \). Aggregating these rates to produce the additional arrival rate of batches of size \( n \) at station \( j \) yields

\[
\begin{align*}
\nu_j^c (n) & = \sum_{i=1}^{\ell(s)} \sum_{s_i = -e_j, i = \ell(\epsilon) - n + 1, \cdots, \ell(s)} \beta_i (i) 1[s_i = -e_j, i = \ell(\epsilon) - n + 1, \cdots, \ell(s)] \\
& = \sum_{m=n+1}^{\infty} \sum_{m=n+1}^{\infty} \mu_i^c (k) p_i (n) \eta(k e_i - (m-n) e_j) \
\end{align*}
\]

Thus, the compensating boundary rates of Miyazawa and Taylor [19] are recovered, and the product form result can now be concluded from Theorem 1.

3.2 Networks with positive and negative signals

Consider a queueing network with \( N \) nodes. There are three types of entities: customers, positive signals, and negative signals, denoted by \( c, c^+ \), and \( c^- \), respectively. Exogenous customers, positive signals, and negative signals arrive at node \( i \) according to Poisson processes with rates \( \nu_i(c^-), \nu_i(c^+) \), and \( \nu_i(c^-) \), respectively. There is a single server at each node and the service times at node \( i \) are exponentially distributed with rate \( \mu_i \). Type \( c \) is a regular customer and its arrival at a node does not induce any instantaneous event (so it simply adds one customer at the arriving node). When a regular customer at node \( i \) completes service, it leaves for node \( j \) as a regular customer with probability \( r_{i(j,c),j}(c) \), a positive signal with probability \( r_{i(j,c),j}(c^+) \), and a negative signal with probability \( r_{i(j,c),j}(c^-) \), and it leaves the network with probability \( r_{i(j,c),0} \). Positive and negative signals induce immediate transitions upon arrival at a node. A positive signal \( c^+ \) arriving at node \( i \) increases the number of customers at node \( i \) by 1, and it leaves immediately for node \( j \) as a regular customer with probability \( r_{i(j,c^+),j}(c) \), as a positive signal with probability \( r_{i(j,c^+),j}(c^+) \), as a negative signal with probability \( r_{i(j,c^+),j}(c^-) \), and it leaves the network with probability \( r_{i(j,c^+),0} \). A negative signal \( c^- \) arriving at node \( i \) decreases the number of customers by 1, as long as there is a customer there upon its arrival, and it then leaves immediately for node \( j \) as a regular customer with probability \( r_{i(j,c^-),j}(c) \), as a positive signal with probability \( r_{i(j,c^-),j}(c^+) \), as a negative signal with probability \( r_{i(j,c^-),j}(c^-) \), and it leaves the network with probability \( r_{i(j,c^-),0} \). A negative signal arriving at an empty node is lost. Obviously,

\[
\sum_{j=1}^{N} r_{i(j,c),j} + \sum_{j=1}^{N} r_{i(j,c^+),j} + \sum_{j=1}^{N} r_{i(j,c^-),j} + r_{i,0} = 1, 
\]

for \( i = 1, 2, \cdots, N \), and \( u \in \{c, c^+, c^-\} \). This network has been studied by Chao and Miyazawa [13].

A transition of this network can consist of any number of customer additions or deletions at different nodes of the network. For example, a sequence \( i_1, i_2, \cdots, i_h \) such that \( r_{i_1(c^-),i_2(c^+)} \)
A string transition for this network is initiated either at a node (by a service completion), or by an exogenous arrival, and consists of sequentially adding or deleting one customer at the nodes. Thus, the set of increment vectors is $\mathcal{A} = \{ \epsilon_i, \ i = 1, 2, \ldots, N \}$.

The rate $\lambda_{s}$ for a string transition $s$ typically consists of an initiating rate, $\nu_i(u)$ for strings initiated by arrivals, and $\mu_i$ for strings initiated by service completions, and a product of routing probabilities $r_{(i,i')}(j,u')$. For example, a string $s$ initiated by the arrival of a positive signal at node $i_1$ that subsequently visits nodes $i_2$ as positive signal, $i_3$ as negative signal, and node $i_4$ as regular customer has rate

$$
\lambda_s = \nu_{i_1}(c+)r_{(i_1,i_2)}(c_2)\cdots r_{(i_4,i_5)}(c_6)\epsilon_{i_5}^{-1}\epsilon_{i_4}^{-1}\epsilon_{i_3}^{-1}.
$$

As a consequence, the rates can be decomposed into an initial segment and a tail segment, where the initial segment contains the initiating rate and a product of routing probabilities, and the tail segment contains a product of routing probabilities, only. For example, for a string $ss'$, initiated by the arrival of a positive signal at node $i_1$,

$$
\lambda_{ss'} = \nu_{i_1}(c+)r_{(i_1,i_2)}(c_2)\cdots r_{(i_4,i_5)}(c_6)\epsilon_{i_5}^{-1}\epsilon_{i_4}^{-1}\epsilon_{i_3}^{-1}r_{(i_6,i_7)}(c_8)\cdots r_{(i_t,i_{t+1})}(c_t)\epsilon_{i_t}^{-1} \epsilon_{i_{t-1}}^{-1} \cdots \epsilon_{i_4}^{-1} \epsilon_{i_3}^{-1} \epsilon_{i_2}^{-1} \cdots \epsilon_{i_1}^{-1}.
$$

where $\lambda_s$ contains the contribution of string $s$, and $\lambda_{s'}$ contains the product of routing probabilities occurring after string $s$:

$$
\hat{\lambda}_s = \nu_{i_1}(c+)r_{(i_1,i_2)}(c_2)\cdots r_{(i_4,i_5)}(c_6)\epsilon_{i_5}^{-1}\epsilon_{i_4}^{-1}\epsilon_{i_3}^{-1}r_{(i_6,i_7)}(c_8)\cdots r_{(i_t,i_{t+1})}(c_t)\epsilon_{i_t}^{-1} \epsilon_{i_{t-1}}^{-1} \cdots \epsilon_{i_4}^{-1} \epsilon_{i_3}^{-1} \epsilon_{i_2}^{-1} \cdots \epsilon_{i_1}^{-1}.
$$

Observe that $\sum_{s'} \lambda_{ss'} = \hat{\lambda}_s$, as $\hat{\lambda}_{s'}$ consists of routing probabilities only. For each string $ss'$ we can thus define decomposed rates $\lambda_{ss'} = \hat{\lambda}_s \hat{\lambda}_{s'}$.

Based on this observation, define the average arrival rate of type $u$ at node $j$:

$$
\alpha_j(u) = \sum_s \eta(s) \hat{\lambda}_s r_{(i_s,j)}(c_{(i_s)},(j,u)).
$$

Using the average arrival rates, the traffic equations (3) can be written as

$$
\alpha_j(c) + \alpha_j(c+) = \sum_s \eta(s) \hat{\lambda}_s \{ r_{(i_s,j)}(c_{(i_s)},(j,c)) + r_{(i_s,j)}(c_{(i_s)},(j,c+)) \}
$$

$$
= \sum_s \eta(s) \hat{\lambda}_{s_{c+}}
$$

$$
= \sum_s \eta(s) \eta(c_j) \hat{\lambda}_s r_{(i_s,j)}(c_{(i_s)},(j,c-)) + \eta(c_j) \mu_j
$$

$$
= \eta(c_j) \{ \alpha_j(c-) + \mu_j \}.
$$

(17)

The average arrival rates are further related as follows. For $a^+ = e_j^{-1}$ representing the arrival of a positive signal at node $j$, for an exponential $\eta$, on the one hand (by separately considering
the contribution \( \nu_j(c+) \) for \( s = 0 \), and the contribution for \( s \neq 0 \) in the summation
\[
\sum_{s,s'} \eta(s) \lambda_{s,s'} = \nu_j(c+) + \sum_{s,s'} \sum_{k=1}^N \eta(s) \eta(c_k^{-1}) \lambda_{s,c_k^{-1}} \eta(c_k^{-1}) 1[c_{j(s)} + 1 = c+]
\]
\[
+ \sum_{s,s'} \sum_{k=1}^N \eta(s) \eta(c_k^{-1}) \lambda_{s,c_k^{-1}} \eta(c_k^{-1}) 1[c_{j(s)} + 2 = c+]
\]
\[
= \nu_j(c+) + \sum_{k=1}^N \sum_{s} \eta(s) \lambda_{s} \eta(c_k^{-1}) \mu \tau_{(i(i), c(i)), (k,c+)} \eta(c_k^{-1}) \tau_{(k,c-), (j,c+)}
\]
\[
+ \sum_{k=1}^N \sum_{s} \eta(s) \lambda_{s} \eta(c_k^{-1}) \mu \tau_{(i(i), c(i)), (k,c+)} \eta(c_k^{-1}) \tau_{(k,c-), (j,c+)}
\]
\[
= \nu_j(c+) + \sum_{k=1}^N \sum_{s} \eta(c_k^{-1}) \alpha_j(c+) \tau_{(k,c+), (j,c+)}
\]
\[
+ \sum_{k=1}^N \sum_{s} \eta(c_k^{-1}) \alpha_j(c+) \tau_{(k,c+), (j,c+)}
\]
where \( c_{j(s)}+2 \) denotes the type arriving at node \( j \), and observing that for a customer to generate a positive signal it must be that service of this customer is the initiating part of the string, and on the other hand, for \( a \) representing the arrival of a positive signal at node \( j \),
\[
\sum_{s,s'} \eta(s) \lambda_{s,s'} = \sum_{s} \eta(s) \lambda_{s} \eta(c_k^{-1}) \mu \tau_{(i(i), c(i)), (k,c+)}
\]
\[
= \alpha_j(c+).
\]
Equating these expressions with \( \eta(x) = \prod_{k=1}^N \rho_k^{x_k} \), and observing that for \( u = c \), and \( u = c- \), similar expressions can easily be obtained following the arguments for \( u = c+ \), we obtain that, for \( u = c, c+, c- \),
\[
\alpha_j(u) = \nu_j(u) + \sum_{k=1}^N \rho_k \mu \tau_{(k,c+), (j,u)} + \sum_{k=1}^N \rho_k \alpha_k(c-) \tau_{(k,c-), (j,u)} + \sum_{k=1}^N \rho_k^{-1} \alpha_k(c+) \tau_{(k,c+), (j,u)}.
\]

The equations (18) form an alternative for the traffic equations (3). For a solution \( \alpha_j(u) \), \( u = c, c+, c- \), of (18) \( \rho_j \) is determined by (17). This approach was taken in Chao and Miyazawa [13], where (18) is defined as traffic equations.

Invoking Theorem 1 requires additional boundary rates compensating for partial strings. Observe that a string can jump out of \( E \) from state \( x = (x_1, x_2, \cdots, x_N) \) such that \( x_i = 0 \) for some \( i \), only. Therefore, by (1) and (2), the compensating rate to the process is introduced only at boundary states \( x \) with \( x_i = 0 \), and the compensating string enters \( E \) from boundary state \( x \) as a positive signal. So, whenever node \( i \) is empty there is an additional arrival rate of positive signals with rate \( \alpha_i(c+) \rho_i^{-1} \), as this rate is the aggregation of the total rate into \( E \) due to positive signals:
\[
\sum_{s} \sum_{j=1}^N \beta_j(s) 1[s_j = -c_i, c_j = c+] = \sum_{s,s'} \lambda_{s,c_i^{-1}} \eta(s) \lambda_{s,c_i^{-1}} 1[c_{j(s)} + 1 = c+]
\]
\[
+ \sum_{s} \eta(s) \lambda_{s} \eta(c_i^{-1}) \mu \tau_{(i(i), c(i), (i,c+))} \eta(c_i^{-1})
\]
\[
= \alpha_i(c+) \rho_i^{-1}.
\]
Theorem 1 implies that the equilibrium distribution of the network is given by

\[
\pi(x) = \prod_{i=1}^{N} (1 - \rho_i) \rho_i^x_i,
\]

which corresponds to the result obtained in Chao and Miyazawa [13].

3.3 Reflecting boundaries

As we discussed in Section 2, an alternative modification at the boundary is to reflect the string transition via \( a \) back into the state space, if vector \( a \) results in an infeasible state. In this section we present a model with reflecting boundaries. For this network, strings \( s \in \mathcal{S} \) are those of the network of Section 2, but the interaction with the boundary is different.

A string \( s \in \mathcal{S} \) can be separated as \( s = s_h a^t s_t \), where \( s_h \) and \( s_t \) denote the head and tail of string \( s \), and \( a \) is the increment vector relating head to tail. Let \( f, g \) be \( 1 \)-\( 1 \) mappings on the heads and tails of the strings, respectively. For string \( s = s_h a^t s_t \in \mathcal{S} \), the mappings \( f \) and \( g \) define a unique string

\[
\delta = \delta_h a^{-t} \delta_t, \quad \text{where} \quad \delta_h = f(s_h), \quad \delta_t = g(s_t),
\]

referred to as the reflected string of string \( s = s_h a^t s_t \). We assume that \( \delta \in \mathcal{S} \).

Similar to the compensating strings of Section 2, reflected strings are used to model the interaction with the boundary. To this end, consider string transition \( s = s_h a^t s_t \) from state \( x \in \mathcal{E} \) such that \( x - s(k) \in \mathcal{E}, k = 1, \ldots, \ell(s_h) \), \( x - s_h - a \notin \mathcal{E} \), i.e., string \( s \) jumps out of \( \mathcal{E} \) with increment vector \( a \). The string transition is reflected back into \( \mathcal{E} \), via the reflected string \( f(s_h) a^{-t} g(s_t) \), to \( x - s_h - g(s_t) \), see Figure 4(a). Thus a string \( s_h g(s_t) \) results as a consequence of string \( s_h a^t s_t \) jumping out of \( \mathcal{E} \) due to increment vector \( a \). It is important that the resulting string \( s_h g(s_t) \) is uniquely determined by the string \( s_h a^t s_t \). This not only implies that the rate for this new string is determined by the rate \( \lambda_a \) for string \( s_h a^t s_t \), but also implies that the string \( s_h a^t s_t \) that starts the string transition also determines the end point (state \( x - s_h - g(s_t) \)) of the string transition.

![Figure 4(a)](image1)

![Figure 4(b)](image2)

The above modeling assumes that the new string \( s_h g(s_t) \) remains inside \( \mathcal{E} \). If this is not the case (i.e., \( g(s_t) \) jumps out of \( \mathcal{E} \), the string is again reflected into \( \mathcal{E} \) via the reflected string for
Reflection of strings continues until the transition terminates when it is mapped into a string such that the tail remains inside \( \mathcal{E} \). See Figure 4(b). For a string that is reflected \( R \) times (\( R = 3 \) in Figure 4(b)), let \( s^{(r)} \) denote the \( r \)-th string in the sequence of reflected strings, \( r = 1, 2, \ldots, R \), and \( s^{(0)} \) be the initial string. Denote \( s^{(r)} = s_{k}^{(r)}(a^{(r)-1})^{-\epsilon^{(r)-1}}s_{k}^{(r)} \), where string \( s^{(r)-1} \) jumps out of \( \mathcal{E} \) due to increment vector \( a^{(r)-1} \), \( r = 1, 2, \ldots, R \). Furthermore, string \( s^{(r)} \) leaves \( \mathcal{E} \) due to its tail segment, \( s_{k}^{(r)} \). Therefore, let the tail of each string be further decomposed as \( s_{k}^{(r)} = s_{i}^{(r)}a^{(r)}s_{k}^{(r)} \). Observe that the strings in the sequence are uniquely determined by the initial string, \( s^{(0)} \), and the increment vectors \( a^{(r)} \) due to which the strings leave \( \mathcal{E} \).

We will assume that for each string the number of reflections until completion of the sequence of strings is finite. Then the rate for a reflected string starting with \( s^{(0)} \) that leaves \( \mathcal{E} \) with increment vector \( a^{(0)} \) is uniquely determined by the rate \( \lambda_{s^{(0)}} \) and the mappings \( f \) and \( g \). This rate is

\[
\tau_{s^{(0)}}(x, y) = \lambda_{s^{(0)}} \left[ \sum_{R=0}^{\infty} \left[ x - \sum_{i=0}^{R-1} s_{i}^{(0)} - s_{k}^{(r)}(k) \in \mathcal{E}, k = 0, \ldots, \ell(s^{(r)}), \right. \right.
\]

\[
\left. x - \sum_{i=0}^{R-1} s_{i}^{(0)} - \epsilon^{(r)}a^{(r)} \notin \mathcal{E}, r = 0, \ldots, R - 1, \right]
\]

\[
y = x - \sum_{i=0}^{R-1} s_{i}^{(0)} - s_{k}^{(R)}, \right] ,
\]

where we have denoted \( s^{(0)} = s_{i}^{(0)} \) to simplify the expression. The contribution for \( R = 0 \) corresponds to a string \( s = s^{(0)} \) that stays inside \( \mathcal{E} \):

\[
\tau_{s}(x, y) = \lambda_{s} 1[x - s(k) \in \mathcal{E}, k = 0, \ldots, \ell(s), y = x - s],
\]

as was used in Section 2. The transition rates are

\[
quadrangle{x, y} = \sum_{s \in S} \tau_{s}(x, y), \quad x, y \in \mathcal{E}.
\]

We have the following result.

**Theorem 2** Let \( \eta : \mathcal{E} \to \mathbb{R}_+ \) be a solution of the traffic equations

\[
\eta(a^{e}) \sum_{s, s'} \eta(s) \lambda_{s a^{e} s'} = \sum_{s, s'} \eta(s) \lambda_{s a^{e-1} s'}, \quad \text{for all } a \in A \setminus \{0\}, \epsilon \in \{-1, 1\}.
\]

Assume that \( \eta \) is exponential, and that, for all \( a \in A, s' \in S \),

\[
\eta(ca) \sum_{s} \eta(s) \lambda_{s a^{e} s'} = \sum_{s} \eta(f(s)) \lambda_{f(s) a^{e-1} g(s')}. \quad \text{(20)}
\]

Then

\[
\pi(x) = B \eta(x), \quad x \in \mathcal{E},
\]

is the unique equilibrium distribution of \( X \), where \( B \) is a normalizing constant.

**Proof** We will show that \( \pi(x) \) satisfies global balance. To this end, observe that, for \( x \in \mathcal{E} \),

\[
\pi(x) \sum_{y \in \mathcal{E}} \eta(x, y) = \sum_{s \in S} \lambda_{s} \eta(x).
\]
For the rate into \( x \), note that the reflected string transition starting with \( s^{(0)} \), with \( R \) reflections, say, gives the following contribution:

\[
\pi(y) r_{s^{(0)}}(y, x) \mathbf{1} \left[ y = z + \sum_{t=0}^{R-1} s_{i_{th}}^{(1)} + s^{(R)} \right] \\
= \pi(y) \lambda_{s_{i_{th}}^{(1)}(a^{(0)}), s_{i_{th}}^{(0)}} \mathbf{1} \left[ y - \sum_{t=0}^{R-1} s_{i_{th}}^{(1)} - s_{i_{th}}^{(0)}(k) \in \mathcal{E}, \ k = 0, \ldots, \ell(s_{i_{th}}^{(0)}) \right] \\
y - \sum_{t=0}^{R-1} s_{i_{th}}^{(1)} - e^{(r)}(a^{(r)}) \notin \mathcal{E}, \ r = 0, \ldots, R - 1, \\
y = z + \sum_{t=0}^{R-1} s_{i_{th}}^{(1)} + s^{(R)} .
\]

(21)

Under the restrictions expressed in the indicator function on the right hand side of (21), observe that

\[
\sum_{s_{i_{th}}^{(0)}(a^{(0)}), s_{i_{th}}^{(0)}} \eta(s_{i_{th}}^{(1)}) \lambda_{s_{i_{th}}^{(0)}(a^{(0)})} s_{i_{th}}^{(0)} \eta(s_{i_{th}}^{(1)}) s_{i_{th}}^{(0)} \lambda_{s_{i_{th}}^{(0)}(a^{(0)})} s_{i_{th}}^{(0)} = \\
\sum_{s_{i_{th}}^{(0)}(a^{(0)}), s_{i_{th}}^{(0)}} \eta(s_{i_{th}}^{(1)}) \lambda_{s_{i_{th}}^{(0)}(a^{(0)})} s_{i_{th}}^{(0)} \lambda_{s_{i_{th}}^{(0)}(a^{(0)})} s_{i_{th}}^{(0)} = \\
\sum_{s_{i_{th}}^{(0)}(a^{(0)}), s_{i_{th}}^{(0)}} \eta(s_{i_{th}}^{(1)}) \lambda_{s_{i_{th}}^{(0)}(a^{(0)})} s_{i_{th}}^{(0)} s_{i_{th}}^{(0)} = \\
\sum_{s_{i_{th}}^{(0)}(a^{(0)}), s_{i_{th}}^{(0)}} \eta(s_{i_{th}}^{(1)}) \lambda_{s_{i_{th}}^{(0)}(a^{(0)})} s_{i_{th}}^{(0)} s_{i_{th}}^{(0)} .
\]

(22)

where we have defined \( s_{i_{th}}^{(1)} := s_{i_{th}}^{(1)}(a^{(0)}) \), \( s_{i_{th}}^{(0)} := s_{i_{th}}^{(0)} \). Observing that \( s_{i_{th}}^{(1)} \) jumps out of \( \mathcal{E} \) due to increment vector \( a^{(1)} \), we may again use (20) to relate string \( s^{(1)} \) to string \( s^{(2)} \).

Iterating this procedure yields, under the restrictions expressed in the indicator function on the right hand side of (21),

\[
\sum_{s_{i_{th}}^{(0)}(a^{(0)}), s_{i_{th}}^{(0)}} \eta \left( \sum_{t=0}^{R-1} s_{i_{th}}^{(1)} + s^{(R)} \right) \lambda_{s_{i_{th}}^{(0)}(a^{(0)})} s_{i_{th}}^{(0)} = \\
\sum_{s_{i_{th}}^{(0)}(a^{(0)}), s_{i_{th}}^{(0)}} \eta \left( s^{(R)}(a^{(R-1)}) \right) \lambda_{s_{i_{th}}^{(0)}(a^{(R-1)})} s_{i_{th}}^{(0)} s_{i_{th}}^{(0)} ,
\]

or, equivalently, under the restrictions expressed in the indicator function on the right hand side of (21),

\[
\sum_{s^{(0)}} \eta \left( \sum_{t=0}^{R-1} s_{i_{th}}^{(1)} + s^{(R)} \right) \lambda_{s^{(0)}} = \sum_{s^{(R)}} \eta(s^{(R)}) \lambda_{s^{(R)}} .
\]

(23)

The right hand side of the global balance equations, \( \sum_{s^{(0)}} \pi(y) q(y, x) \), can be decomposed into a sum over the strings that end in \( x \). Such strings are either complete strings (that have stayed inside \( \mathcal{E} \)), or reflected strings for which the last segment, \( s^{(R)} \), ends in \( x \). For this decomposition the 1-1 relation between \( s^{(R)} \) and \( s^{(0)} \) via the unicity of reflected strings plays a crucial role.
We thus obtain
\[
\sum_{y \in E} \pi(y)q(y,x) = \sum_{y \in E} \sum_{s \in S} \sum_{R=0}^{\infty} \pi(y) \tau_s(y,x) \mathbb{1}[s^{(R)} = s] \frac{1}{\pi(x) \sum_{s \in S} \eta(s^{(R)}\lambda_s) \mathbb{1}[s^{(R)} = s] - \pi(x) \sum_{s \in S} \eta(s)\lambda_s},
\]
observing that the number of reflections $R$, and all strings $s^{(r)}$, $r = 0, \cdots, R$, in a sequence of reflected strings are uniquely determined by the string $s = s^{(R)}$ that ends in $x$, and therefore that also the state $y \in E$ in $\tau_s(y,x) \mathbb{1}[s^{(R)} = s]$ is uniquely determined by $s \in S$. Invoking (21) and (22) we then obtain that
\[
\sum_{y \in E} \pi(y)q(y,x) = \pi(x) \sum_{s \in S} \eta(s^{(R)}\lambda_s) \mathbb{1}[s^{(R)} = s] = \pi(x) \sum_{s \in S} \eta(s)\lambda_s.
\]
Relation (7) completes the proof.

The model with reflecting boundaries is closely related to the model of Section 2. With reflecting boundaries, each partial string has a unique continuation via the reflected string. In Section 2 a compensating string is randomly selected out of all possible continuations of strings jumping out of $E$. Random selection is made possible (or is even required) in Section 2, because the continuation via a compensating string is not part of the transition resulting in a partial string. Therefore, due to the Markov property, the information of the (partial) string that has taken $X$ to a boundary state is lost. Random selection of compensating rates requires (4), whereas reflection requires (20). Thus the reflected boundary model provides additional motivation for the compensating rates of Section 2.

An alternative method of reflecting strings is obtained by defining the reflected string, $\tilde{s}$, of a string $s = s_ha\tilde{s}_t$, as
\[
\tilde{s} = \tilde{s}_ha^{-\epsilon}\tilde{s}_t, \quad \text{where} \quad \tilde{s}_h = \hat{f}(s_h), \quad \tilde{s}_t = \hat{g}(s_t),
\]
for 1-1 mappings $\hat{f}, \hat{g}$, i.e., relating the tail of the reflected string to the head of the original string. This requires an additional assumption
\[
\eta'(s)\eta(s)\lambda_{sa^{\epsilon}a} = \eta(\hat{f}(s'))\lambda_{f(s')a^{\epsilon}\hat{g}(s)}
\]
in Theorem 2. This restriction also forms a decomposition of the traffic equations relating the rates for string $s$ using increment vector $a$ to the rate for the reflected string depositing increment vector $a$.

Another alternative for reflecting strings upon each crossing of the boundary is provided by the reflection scheme in which only the tail segment is reflected if a string crosses the boundary an odd number of times. In that case the tail segment leaves the state space and is reflected into the state space by one of the reflections described above. Note that a string that crosses the boundary an even number of times must start and end inside the state space. From the proof of Theorem 2 it is obvious that this reflection scheme also results in a product form equilibrium distribution.

References