A functional analytic approach towards nonlinear dissipative well-posed systems

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Abstract
The aim of this paper is to develop a functional analytic approach towards nonlinear systems. For linear systems this is well known and the resulting class of well-posed and regular linear systems is well studied. Our approach is based on the theory of nonlinear semigroup and we explain it by means of an example, namely equations of quasi-hyperbolic type.

1 Equations of quasi-hyperbolic type

Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly monotone decreasing surjective function with $\psi(0) = 0$. Associated to this function, we consider the following partial differential equation on the spatial interval $[0, 1]$,

$$\frac{\partial f(t, \eta)}{\partial t}(t, \eta) = \frac{\partial \psi(f)}{\partial \eta}(t, \eta), \quad 0 < \eta < 1, t > 0;$$

$$f(0, \eta) = f_0(\eta), \quad 0 < \eta < 1,$$

$$f(t, 0) = 0, \quad t > 0.$$ (1)

As output we define

$$y(t) = \psi(f(t, 1)).$$ (2)

We want to show that (1)–(2) defines a dissipative system. In particular, this will imply that the output $y$ is a function in $L^1(0, \infty)$.

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As state space we choose \( X = L^1(0,1) \) and we choose \( L^1_{\text{loc}}(0,\infty) \) as our space of output signals. In [4, p.110] it is shown that \( A : D(A) \subset L^1(0,1) \to L^1(0,1) \), given by
\[
(Af)(\eta) := \frac{d\psi(f(\eta))}{d\eta}, \quad f \in D(A),
\]
\[D(A) := \{ f \in C[0,1] \mid f(0) = 0, \psi(f) \text{ is absolutely continuous on } [0,1] \}
\]
is an m-dissipative operator on \( X \). Associated to this m-dissipative operator we can define a semigroup \((T(t))_{t \geq 0}\) of nonlinear contractions on \( X \). This semigroup is defined as
\[
T(t)f_0 = \lim_{\lambda \downarrow 0}(I - \lambda A)^{-[t/\lambda]}f_0. \quad (3)
\]
Unfortunately, in general \( A \) is not the infinitesimal generator of this semigroup, see [4, p. 104]. However, if \( f \) is a classical solution of the Cauchy problem \( \dot{f} = Af \), \( f(0) = f_0 \), then \( f(t) = T(t)f_0 \). Furthermore, if \( f_0 \in D(A) \) and \((T(t)f_0)_{t \geq 0}\) is strongly differentiable a.e., then this function is the unique solution of the Cauchy problem \( \dot{f} = Af \), \( f(0) = f_0 \). In general, \((T(t)f_0)_{t \geq 0}\) only is an integral solution.

Using the method of characteristics, it can be shown that for smooth functions \( f_0 \) the partial differential equation (1) has on some interval \([0,t_0]\) a classical solution. This solution equals \( T(t)f_0 \) for \( t \in [0,t_0] \), [2, p. 189]. Thus the output — as given in (2) — is well-defined on some interval \([0,t_0]\) provided \( f_0 \) is smooth. Dissipativity of a system now guarantees \( L^1 \)-outputs and that the system generates no energy.

**Definition 1.1** We say that the system (1)–(2) is dissipative, if there exists a mapping \( C : L^1(0,1) \to L^1(0,\infty) \) satisfying

1. If for \( f_0 \in L^1(0,1) \) and \( t_0 > 0 \) the partial differential equation (1) possesses a classical solution on \([0,t_0]\), then \((Cf_0)(t) = \psi((T(t)f_0)(1)) \) for \( t \in [0,t_0] \).

2. For any \( t, s \geq 0 \) and every \( f_0 \in L^1(0,1) \) we have
\[
P_{t+s}Cf_0 = P_tCf_0 + S_r(s)P_tC(T(s)f_0).
\]

3. For \( f_0, f_1 \in L^1(0,1) \) and for every \( t_0 > 0 \) we have
\[
\|Cf_0 - Cf_1\|_{L^1(0,t_0)} \leq \|f_0 - f_1\|_{L^1(0,1)} + \|T(t_0)f_0 - T(t_0)f_1\|_{L^1(0,1)}. \quad (4)
\]

Here \( P_t \in \mathcal{L}(L^1(0,\infty)) \) is the projection onto the interval \([0,t]\) and \( S_r(t) \in \mathcal{L}(L^1(0,\infty)) \) is the right-shift by \( t \). As mentioned above, if \( f_0 \) is smooth then there exists an interval on which (1) has a classical solution. On this interval this classical solution equals \( T(t)f_0 \). Hence from item 1. and equation (2) we see that \( y \) equals \( Cf_0 \) on this time-interval. For general \( f_0 \) we call the function \( Cf_0 \) the "generalized" output of system (1)–(2). The main result of this article is as follows.
Theorem 1.2 System (1)–(2) is dissipative.

The proof of this theorem will be given at the end of the section. The following lemma will be useful.

Lemma 1.3 Let \( \phi \) be a strictly monotone decreasing surjective function from \( \mathbb{R} \) to \( \mathbb{R} \) with \( \phi(0) = 0 \), let \( -\infty \leq a < b \leq \infty \) and let \( A_\phi : D(A_\phi) \subset L^1(a,b) \to L^1(a,b) \) be defined by

\[
(A_\phi f)(\eta) := \frac{d\phi(f(\eta))}{d\eta}, \quad f \in D(A_\phi),
\]

\( D(A_\phi) := \{ f \in L^1(a,b) \mid \phi(f) \text{ is absolutely continuous on } [a,b] \text{ and } \phi'(f) \in L^1(a,b) \} \).

Then we have

\[
\|f_1 - f_2 - \lambda(A_\phi f_1 - A_\phi f_2)\|_{L^1(a,b)} \geq \|f_1 - f_2\|_{L^1(a,b)} + \lambda|\phi(f_1(b)) - \phi(f_2(b))| - \lambda|\phi(f_1(a)) - \phi(f_2(a))|.
\]

for all \( \lambda > 0 \) and all \( f_1, f_2 \in D(A) \).

If \( a = -\infty \), then \( f_1(a) = f_2(a) = 0 \) in the above inequality. A similar remark holds if \( b = \infty \).

The proof of this lemma follows largely [4, page 110–111].

Proof: We define the functions \( p_n : \mathbb{R} \to \mathbb{R}, n \in \mathbb{N}, \) by

\[
p_n(\eta) := \begin{cases} -n\eta & |\eta| \leq 1/n \\ -\text{sign } \eta & |\eta| > 1/n \end{cases}.
\]

It is easy to see that the functions \( p_n \) are monotone decreasing, Lipschitz continuous, \( p_n(0) = 0 \) and \( |p_n(\eta)| \leq 1 \) for every \( \eta \in \mathbb{R} \) and every \( n \in \mathbb{N} \). Let \( f_1, f_2 \in D(A) \) and \( \lambda > 0 \) be arbitrarily. Then we have for every \( n \in \mathbb{N} \)

\[
\int_a^b (A_1 f_1 - A_2 f_2) p_n(\phi(f_1(\eta)) - \phi(f_2(\eta))) \, d\eta
\]

\[
= \int_a^b (\phi(f_1(\eta)) - \phi(f_2(\eta)))' p_n(\phi(f_1(\eta)) - \phi(f_2(\eta))) \, d\eta
\]

\[
= \int_a^b \frac{d}{d\eta} (q_n(\phi(f_1(\eta)) - \phi(f_2(\eta)))) \, d\eta
\]

\[
= q_n(\phi(f_1(b)) - \phi(f_2(b))) - q_n(\phi(f_1(a)) - \phi(f_2(a))),
\]
where \( q_n(s) := \int_0^s p_n(\eta) \, d\eta \), and thus
\[
\int_a^b |f_1 - f_2 - \lambda(Af_1 - Af_2)| \, d\eta \\
\geq \int_a^b |f_1 - f_2 - \lambda(Af_1 - Af_2)| \, |p_n(\phi(f_1) - \phi(f_2))| \, d\eta \\
\geq \int_a^b (f_1 - f_2 - \lambda(Af_1 - Af_2)) \, p_n(\phi(f_1) - \phi(f_2)) \, d\eta
\]
\[
\geq \int_a^b (f_1(\eta) - f_2(\eta)) \, p_n(\phi(f_1(\eta)) - \phi(f_2(\eta))) \, d\eta \\
+ \lambda q_n(\phi(f_1(a)) - \phi(f_2(a))) - \lambda q_n(\phi(f_1(b)) - \phi(f_2(b))).
\]

Now \( p_n \) converges pointwise to \( p(\eta) = -\text{sign} \, \eta \) and thus by Lebesgue Dominated Convergence Theorem we finally have
\[
\int_a^b |f_1 - f_2 - \lambda(Af_1 - Af_2)| \, d\eta \\
\geq \int_a^b |f_1(\eta) - f_2(\eta)| \, d\eta + \lambda |\phi(f_1(b)) - \phi(f_2(b))| - \lambda |\phi(f_1(a)) - \phi(f_2(a))|.
\]

In the last step we used that \( \phi \) is strictly monotone decreasing. \[ \square \]

Now we define the mapping \( A_e : D(A_e) \subset X \times L^1(0, \infty) \to X \times L^1(0, \infty) \) by
\[
A_e \left( \begin{pmatrix} f \\ y' \end{pmatrix} \right) = \begin{pmatrix} Af \\ -y' \end{pmatrix}, \quad (5)
\]
\[
D(A_e) = \left\{ \left( \begin{pmatrix} f \\ y \end{pmatrix} \right) \in D(A) \times W^{1,1}(0, \infty) \mid y(0) = \psi(f(1)) \right\}. \quad (6)
\]

This operator is m-dissipative on \( L^1(0, 1) \times L^1(0, \infty) \), where the norm on this product space is the sum of the separate norms.

**Theorem 1.4** The operator \( A_e \) as defined in (5) and (6) is m-dissipative on \( L^1(0, 1) \times L^1(0, \infty) \).

**Proof** First we prove that \( A_e \) is dissipative, that is, for all \( \lambda > 0 \) and all \( z_1, z_2 \in D(A_e) \) we have
\[
\|z_1 - z_2 - \lambda(A_e z_1 - A_e z_2)\| \geq \|z_1 - z_2\|.
\]
Thus let $\lambda > 0$ and $(\frac{f_1}{y_1}), (\frac{f_2}{y_2}) \in D(A_e)$ be arbitrarily. Applying Lemma 1.3 to the operator $A$ and noting that $f_1, f_2 \in D(A)$ we get

$$
\|f_1 - f_2 - \lambda (Af_1 - Af_2)\|_{L^1(0,1)} \\
\geq \|f_1 - f_2\|_{L^1(0,1)} + \lambda|\psi(f_1(1)) - \psi(f_2(1))| - \lambda|\psi(f_1(0)) - \psi(f_2(0))| \\
= \|f_1 - f_2\|_{L^1(0,1)} + \lambda|y_1(0) - y_2(0)|
$$

(7)

since $f_1, f_2 \in D(A)$ and $(\frac{f_1}{y_1}), (\frac{f_2}{y_2}) \in D(A_e)$. Choosing $a = 0, b = \infty$, $\phi(s) := -s$, $s \in \mathbb{R}$ in Lemma 1.3 and noting that $y_1, y_2 \in W^{1,1}(0,\infty)$ we get

$$
\|y_1 - y_2 - \lambda(-\dot{y}_1 + \dot{y}_2)\|_{L^1(0,\infty)} \geq \|y_1 - y_2\|_{L^1(0,\infty)} - \lambda|y_1(0) - y_2(0)|.
$$

(8)

Finally, combining (7) and (8) gives for $(\frac{f_1}{y_1}), (\frac{f_2}{y_2}) \in D(A_e)$

$$
\|(\frac{f_1}{y_1}) - (\frac{f_2}{y_2}) - \lambda (A_e (\frac{f_1}{y_1}) - A_e (\frac{f_2}{y_2}))\| \\
= \|f_1 - f_2 - \lambda (Af_1 - Af_2)\|_{L^1(0,1)} + \|y_1 - y_2 - \lambda(-\dot{y}_1 + \dot{y}_2)\|_{L^1(0,\infty)} \\
\geq \|f_1 - f_2\|_{L^1(0,1)} + \|y_1 - y_2\|_{L^1(0,\infty)} \\
= \|(\frac{f_1}{y_1}) - (\frac{f_2}{y_2})\|.
$$

Thus $A_e$ is dissipative on $L^1(0,1) \times L^1(0,\infty)$.

In order to show that $A_e$ is m-dissipative on $L^1(0,1) \times L^1(0,\infty)$ it remains to prove that the range of $I - A_e$ equals $L^1(0,1) \times L^1(0,\infty)$. Let $(\frac{f}{y}) \in L^1(0,1) \times L^1(0,\infty)$ be arbitrarily. Since $A$ is m-dissipative on $L^1(0,1)$, see [4], we have that the range of $(I - A)$ equals $L^1(0,1)$. Thus there exists $f \in D(A)$ such that

$$(I - A)f = \tilde{f}.$$

We now define $y \in L^1(0,\infty)$ by

$$
y(t) := e^{-t}\psi(f(1)) + \int_0^t e^{-(t-s)}\tilde{y}(s) \, ds.
$$

It is easy to see that

$$
y(t) + \dot{y}(t) = \tilde{y}(t) \quad \text{and} \quad y(0) = \psi(f(1)).
$$

Thus $(\frac{f}{y}) \in D(A_e)$ and $(I - A_e)(\frac{f}{y}) = (\frac{\tilde{f}}{\tilde{y}})$. This proves that $A_e$ is m-dissipative on $L^1(0,1) \times L^1(0,\infty)$. This completes the proof.

The following lemma is easy to verify.

**Lemma 1.5** Let $(\frac{f_0}{y_0}) \in D(A_e)$ and $t_0 > 0$. Then the partial differential equation (1) possesses a classical solution on the interval $t \in [0, t_0]$ if and only if the Cauchy problem

$$
\frac{d}{dt} \begin{pmatrix} f(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} Af(t) \\ -\frac{A}{y_0}(y(t)) \end{pmatrix}
$$

has a classical solution on the interval $[0, t_0]$. 

5
Let $T_e(t)$ be the semigroup associated to $A_e$, i.e.,
\[ T_e(t) \left( \begin{array}{c} f_0 \\ y_0 \end{array} \right) = \lim_{\lambda \to 0} (I - \lambda A_e)^{-\frac{t}{\lambda}} \left( \begin{array}{c} f_0 \\ y_0 \end{array} \right). \tag{9} \]
and let $T(t)$ be the semigroup associated to $A$, see (3). An easy calculation shows that for $\lambda > 0$ and \( \left( \begin{array}{c} f_0 \\ y_0 \end{array} \right) \in L^1(0, 1) \times L^1(0, \infty) \) we have
\[ (I - \lambda A_e)^{-1} \left( \begin{array}{c} f_0 \\ y_0 \end{array} \right) = \left( e^{-\lambda \psi((I - \lambda A)^{-1} f_0)}(1) + (I - \lambda D_R)^{-1} y_0 \right), \]
where $D_R f := -f'$. Thus similar as the non-linear operator $A_e$, the resolvent operator has the nice property that
\[ (I - \lambda A_e)^{-1} \left( \begin{array}{c} f_0 \\ y_0 \end{array} \right) = (I - \lambda A_e)^{-1} \left( \begin{array}{c} f_0 \\ 0 \end{array} \right) + (I - \lambda A_e)^{-1} \left( \begin{array}{c} 0 \\ y_0 \end{array} \right). \]
Furthermore, since $D_R$ is a linear operator, the resolvent is linear in the second component. Using this, equation (9), (3), and the fact that $D_R$ is the infinitesimal generator of the right-shift semigroup $(S_r(t))_{t \geq 0}$ we obtain that $T_e(t)$ is given by
\[ T_e(t) \left( \begin{array}{c} f_0 \\ y_0 \end{array} \right) = \left( \begin{array}{c} T(t) f_0 \\ Q(t) f_0 + S_r(t) y_0 \end{array} \right) \tag{10} \]
for some mapping $Q_t$ from $L^1(0, 1)$ to $L^1(0, \infty)$. Since $(T_e(t))_{t \geq 0}$ is a contraction semigroup, we have
\[ \|Q_t f_0 - Q_t f_1\|_{L^1(0, \infty)} \leq \|f_0 - f_1\|_{L^1(0, 1)} - \|T(t) f_0 - T(t) f_1\|_{L^1(0, 1)} \tag{11} \]
Moreover, the semigroup property implies
\[ Q_{t+s} f_0 = Q_t T(s) f_0 + S_r(t) Q_s f_0 \tag{12} \]

**Theorem 1.6** For \( \left( \begin{array}{c} f_0 \\ y_0 \end{array} \right) \in D(A_e) \) such that the partial differential equation (1) possesses a classical solution on some interval $[0, t_0]$ we have that for $t \in [0, t_0]$
\[ \left( Q_t f_0 \right)(\rho) = \begin{cases} \psi((T(t - \rho) f_0)(1)), & \text{for } \rho \in [0, t] \\ 0 & \text{for } \rho > t. \end{cases} \]

**Proof** Let \( \left( \begin{array}{c} f_0 \\ y_0 \end{array} \right) \in D(A_e) \) arbitrarily such that the partial differential equation (1) possesses a classical solution on some interval $[0, t_0]$. Then \( \left( \begin{array}{c} f(t) \\ y(t) \end{array} \right) := T_e(t) \left( \begin{array}{c} f_0 \\ y_0 \end{array} \right) \) is a classical solution of $\dot{z}(t) = A_e z(t)$ on $[0, t_0]$ and thus $f(t) = T(t) f_0$. The second equation equals the partial differential equation
\[ \frac{\partial y(t)(s)}{\partial t} = -\frac{\partial y(t)(s)}{\partial s}, \quad 0 \leq t \leq t_0, \quad s \geq 0. \]
with initial condition \((y(0)) (s) = y_0(s)\). The solution equals \((y(t)) (s) = g(t - s), s > 0 \text{ and } t \in [0, t_0],\) for some function \(g \in W^{1,1}(-\infty, t_0)\) with \(g(-s) = y_0(s)\) for \(s > 0\). Since \(\begin{pmatrix} f(t) \\ y(t) \end{pmatrix} \in D(A_e),\) we see that \((y(t))(0) = \psi((f(t))(1))\) for \(t \in [0, t_0].\) Thus
\[
    g(t) = \psi((f(t))(1)), \quad t \in [0, t_0].
\]
By equation (10) we have that \(g(t) = Q_t f_0 + S_e(t)y_0.\) Combining this with the results above, implies that for \(s \in [0, t]\) and \(t \leq t_0\)
\[
    \psi((f(t-s))(1)) = g(t-s) = (y(t)) (s)
    = (Q_t f_0 + S_e(t)y_0) (s) = (Q_t f_0) (s).
\]
For \(s > t\) and \(t \leq t_0,\) we have that
\[
y_0(s-t) = g(t-s) = (y(t)) (s)
    = (Q_t f_0 + S_e(t)y_0) (s) = (Q_t f_0) (s) + y_0(s-t).
\]
This implies that \((Q_t f_0) (s) = 0\) for \(s > t,\) whenever \(t \in [0, t_0].\) Combining this with equation (13) gives the expression for \(Q_t.\)

**Lemma 1.7** If \(f_0 \in L^1(0, 1)\) and \(t > 0,\) then we have \((Q_t f_0) (\rho) = 0\) for a.e. \(\rho > t.\)

**Proof:** Let \(f_0 \in L^1(0, 1)\) and \(t > 0\) be arbitrarily. Then definitions of \(A\) and \(A_e\) imply
\[
(I - \lambda A)^{-1} 0 = 0 \text{ and } (I - \lambda A_e)^{-1} (0) = (0)
\]
for every \(\lambda > 0.\) This together with the fact that \(A_e\) is \(m\)-dissipative shows
\[
\|(I - \lambda A_e)^{-1} (g)\| \leq \|(g)\|, \quad \lambda > 0, f \in L^1(0, 1), y \in L^1(0, \infty). \tag{15}
\]
Moreover, for \((\frac{f}{y}) \in L^1(0, 1) \times L^2(0, \infty)\) we have
\[
(I - \lambda A_e)^{-1} \begin{pmatrix} f \\ y \end{pmatrix} = \left( e^{-\lambda} \Psi\left( ((I - \lambda A)^{-1} f)(1) \right) + (I - \lambda D_R)^{-1} y \right)
\]
and
\[
(I - \lambda A_e)^{-n} \begin{pmatrix} f \\ y \end{pmatrix} = \left( e^{-\lambda} \sum_{k=0}^{n-1} \frac{(\lambda)^k}{k!} \Psi\left( ((I - \lambda A)^{-k} f)(1) \right) + (I - \lambda D_R)^{-n} y \right).
\]
We note, that \((I - \lambda D_R)^{-k} e^{-\lambda} = \frac{(\lambda)^k}{k!} e^{-\lambda}.\) Thus equation (15) implies
\[
\|f_0\|_{L^1(0,1)} \geq \|(I - \lambda A)^{-1} f_0\|_{L^1(0,1)} + \lambda |\Psi((I - \lambda A)^{-1} f_0)(1)| \tag{15}
\]
\[
\geq \|(I - \lambda A)^{-2} f_0\|_{L^1(0,1)} + \lambda |\Psi((I - \lambda A)^{-2} f_0)(1)|
    + \lambda |\Psi((I - \lambda A)^{-1} f_0)(1)|
\]
\[
\geq \|(I - \lambda A)^{-n} f_0\|_{L^1(0,1)} + \lambda \sum_{j=1}^{n} |\Psi((I - \lambda A)^{-j} f_0)(1)|.
\]
In particular, for $\lambda = t/n$ we get
\[ \frac{t}{n} \sum_{j=1}^{n} |\Psi((I - \lambda A)^{-j} f_0)(1)| \leq \|f_0\|. \] (16)

We now define the function $g_n \in L^1(0, \infty)$ by
\[ g_n(s) = e^{-\frac{sn}{t}} \sum_{k=0}^{n-1} \frac{n^k s^k}{k! t^k} \Psi(((I - \frac{t}{n} A)^{-(n-k)} f_0)(1)). \]

We know already that
\[ \lim_{n \to \infty} \|g_n - Q_t f_0\|_{L^1(0, \infty)} = 0. \]

Thus by Lebesgue’s theorem it remain to show that
\[ g_n(s) \to 0 \text{ for almost every } s > t. \]

For $s > t$ we have
\[
|g_n(s)| \leq e^{-\frac{sn}{t}} \sum_{k=0}^{n-1} \frac{n^k s^k}{k! t^k} \Psi(((I - \frac{t}{n} A)^{-(n-k)} f_0)(1))
\leq e^{-\frac{sn}{t}} \sup_{k=0}^{n-1} \left( \frac{n^k s^k}{k! t^k} \right) \sum_{k=0}^{n-1} \Psi(((I - \frac{t}{n} A)^{-(n-k)})
\leq \|f_0\| \frac{n}{t} e^{-\frac{sn}{t}} \frac{n^{n-1} s^n}{n! t^n} \text{ since } s > t
\leq \|f_0\| \frac{n}{s} e^{-\frac{sn}{t}} \frac{n^n s^n}{n! t^n}
\leq \frac{\|f_0\|}{\sqrt{2\pi s}} e^{-\frac{n^2}{2t}} \left( \frac{s}{t} \right)^n \text{ (by Stirlings formula } n^n/n! \leq e^n/\sqrt{2\pi n})
\leq \frac{\|f_0\|}{\sqrt{2\pi s}} \sqrt{n} e^{n} e^{-\frac{s}{t}} \left( \frac{s}{t} \right)^n
\to 0 \text{ for } n \to \infty \text{ since } s > t. \]

\textbf{Proof of Theorem 1.2:} We define the mapping $C_t$ as
\[ C_t = R(t) Q_t, \] (17)
where $R(t) \in L(L^1(0, \infty))$ is defined by $(R(t)f)(s) = f(t - s)$ if $0 \leq s \leq t$ and $(R(t)f)(s) = 0$, otherwise. Since $Q_t$ has its support in $[0, t]$, we have that $Q_t = R(t) C_t$. Using this and (12) we find that
\[ C_{t+s} f_0 = C_s f_0 + S_r(s) C_t T(s) f_0. \]
Since by the definition of $C_t$ we have that $P_tC_t = C_t$ we have that the above equality is equivalent to

$$P_{t+s}C_{t+s}f_0 = P_tC_s f_0 + S_r(s)P_tC_t T(s)f_0. \tag{18}$$

In particular, this shows $(C_{t+s}f_0)(\rho) = (C_sf_0)(\rho)$, a.e. $\rho \in [0, s]$ for any $t, s > 0$. Thus we can define $C : L^1(0, 1) \to L^1(0, \infty)$ by

$$(Cf_0)(\rho) = (C_t f_0)(\rho), \quad t > \rho. \tag{19}$$

Using this and equations (17), (11) it is easy to see that

$$\|Cf_0 - Cf_1\|_{L^1(0, t)} \leq \|f_0 - f_1\|_{L^1(0, 1)} - \|T(t)f_0 - T(t)f_1\|_{L^1(0, 1)}. \tag{20}$$

Hence we have shown that item 2. and 3. of Definition 1.1 hold. In Theorem 1.6 we have shown that it also satisfies item 1. Thus our system is dissipative.

\section*{References}


