PASSIVATION OF UNDERACTUATED SYSTEMS WITH PHYSICAL DAMPING

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Abstract: In recent works, IDA-PBC has been successfully applied to mechanical control problems with no physical damping present. In some cases, the friction terms can be obviated without compromising stability in closed loop. However, in methods that modify the kinetic energy, a controller designed for stabilizing the undamped system might lose passivity, a key property for stabilization, when damping is introduced. This paper presents a necessary and sufficient condition, namely the dissipation condition, for recovering passivity (and hence stability) in such cases. If the dissipation condition is fulfilled, an IDA-PBC redesign is necessary in general, and with this goal two different methods for passivating the damped system are presented.

Keywords: Nonlinear Control, Hamiltonian Systems, Passivity Based Control

1. INTRODUCTION

Recent works in the field of underactuated control have commonly neglected a fundamental issue in modeling and control as is physical damping. In underactuated problems it is intuitive to see that the effects caused by friction in certain directions lie outside the reach of the controller and cannot be compensated for (Ortega et al., 2002). Possibly because of this, friction terms have been repeatedly left unmatched and the classical approach reduces to solve the control problem for an undamped open loop model, and stay in the naïve hope that physical dissipation will help in some way to reach the desired equilibrium point. Nevertheless, it has been proved that in control methods that modify the kinetic energy, such as IDA-PBC (Van der Schaft, 2000) and Controlled Lagrangians (A. Bloch and Marsden, 2000), unmodeled physical damping can cause instability. In some cases, not even the tangent linearization at the equilibrium point preserves stability after the introduction of open loop damping (see (Reddy et al., 2004)).

However, under certain conditions, the knowledge of such friction terms can be exploited to transform the damping terms in the actuated directions in such a way that passivity of the whole system is recovered. In these cases, physical damping actually plays in favor of closed loop dissipation and convergence to the equilibrium. This implies that all results on local and non local stability for the undamped model are preserved. Moreover, in underactuated systems, exploiting physical friction is the only way to obtain strong dissipation (full rank closed loop dissipation matrix), a sufficient condition for local exponential stability at the desired equilibrium point.

2. PROBLEM STATEMENT

The method IDA-PBC for mechanical systems aims at passivating an open loop model of the form

$$\Sigma_1(M,V,G,R) : \begin{bmatrix} \dot{q} \\ p \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & -R(q) \end{bmatrix} \begin{bmatrix} \partial H/\partial q \\ \partial H/\partial p \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} u,$$

where \( q \in \mathbb{R}^n \) are the generalized coordinates and \( p \in \mathbb{R}^n \) the momenta, defined as \( p = M \dot{q} \); \( M \) is the open loop
Hamiltonian is defined as inertia matrix, $R > 0$ the physical damping matrix and the Hamiltonian is defined as

$$H = \frac{1}{2} p^T M^{-1}(q) p + V(q).$$

Assume $G = G(q)$ has constant rank $m < n$ and hence a matrix $G^\perp$ of row rank $n - m$ exists such that

$$G^\perp G = 0, \quad \text{rank}[G^\perp G^\top] = n.$$

In (Ortega and Spong, 2000; Ortega et al., 2002; Gómez-Estern et al., 2001) the IDA-PBC control problem is solved for a class of energy preserving open loop models (of the form $\Sigma(M, V, G, 0)$). In those papers, control laws are designed to transform $\Sigma_1$ into a closed loop Hamiltonian system with dissipation of the form

$$\Sigma_2(M_d, V_d, J_d, G, R_d) :$$

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = (J_d(q,p) - R_d(q,p)) \begin{bmatrix} \frac{\partial H_d}{\partial q} \\ \frac{\partial H_d}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} v,$$

with $J_d$ skew-symmetric and $R_d \geq 0$. The closed loop Hamiltonian dynamics are obtained by setting

$$J_d = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_dM^{-1} & J_2(q,p) \end{bmatrix}, \quad R_d = \begin{bmatrix} 0 & 0 \\ 0 & R_2(q) \end{bmatrix},$$

with $J_2 = -J_2^\perp$, and then equating the open loop and closed loop state equations to solve the set of PDEs in the non-actuated space

$$G^\perp \{ \nabla_q H + R \nabla_p H - M_dM^{-1} \nabla_q H_d + J_2M_d^{-1} p \} = 0.$$  \hspace{1cm} (4)$$

The usual approach assumes $R = 0$ (undamped open loop model), allowing us to split (4) into $p$-dependent (quadratic on $p$) and $p$-independent terms, giving rise to the kinetic and potential energy shaping equations, namely

$$G^\perp \{ \nabla_q (p^T M^{-1} p) - M_dM^{-1} \nabla_q (p^T M_d^{-1} p) \\ + 2(J_2 - R_2)M_d^{-1} p \} = 0$$

$$G^\perp \{ \nabla_q V - M_dM^{-1} \nabla_q V_d \} = 0$$

and $R_2$ is introduced in the subsequent damping injection step. However if $R \neq 0$ we have a third set of matching equations containing new terms that are linear in $p$, that is

$$G^\perp \{ RM^{-1} p + (J_2 - R_2)M_d^{-1} p \} = 0,$$

where $J_2$ is a new design parameter that can be introduced by just splitting the free matrix $J_2$ in terms of the dependence on $p$ as

$$J_2 = J_20(q) + J_21(q,p).$$

If this third matching equation is neglected, the closed loop system may loose passivity and stability. Besides, the existence of a physical damping matrix $R$ is related, as will be seen, with the following useful property:

**Definition 1.** (Strong dissipation). A Hamiltonian system defined on an open set $\{ q \in X \subset \mathbb{R}^n, p \in \mathbb{R}^n \}$ of the form $\Sigma_2$ from (2) with $R_2$ in the form (3), is said to be strongly dissipative if $R_2(q) > 0 \forall q \in X$. For such systems it is easy to check that there is a positive function $\alpha(q,p) > 0$ such that the rate of dissipation is

$$\dot{H}_d = - \left( \frac{\partial H_d}{\partial p} \right)^\top R_2 \frac{\partial H_d}{\partial p} < -\alpha(q,p) \| p \|^2$$

provided that $M_d(q) > 0$ in the domain of interest. This property is not as strong as the existence of a strict Lyapunov function but it is extremely useful for stability analysis. As we will see in the following sections, strong dissipation in IDA-PBC controlled underactuated systems can only be achieved with the aid of physical damping.

3. MAIN RESULT

In this section we will deal with four systems; $\Sigma_1$ as defined in (1), $\Sigma_2$ from (2) and the following two

$$\Sigma_3 = \Sigma_1(M, V, G, 0) \quad \text{Undamped open loop system}$$

$$\Sigma_4 = \Sigma_2(M_d, V_d, J_d, G, 0) \quad \text{Undamped closed loop system}$$

In order to obtain closed loop stability, the IDA-PBC method builds a closed loop system that is passive with storage function the desired closed loop Hamiltonian $H_d$. In recent papers on underactuated control, IDA-PBC control laws were provided such that the undamped system becomes passive after the application of an energy shaping control law $u = u_{es} + v$, with input $v$, storage function

$$H_d = \frac{1}{2} p^T M_d^{-1}(q) p + V_d(q)$$

and passive output $y = G^\top M_d^{-1} p$, i.e.

$$\dot{H}_d < v^\top y.$$  

If we apply the same control law $u = u_{es} + v$ to the physically damped system $\Sigma_1$ we will unlikely obtain a passive closed loop system respect to the triplet $(H_d, v, y)$. Instead of searching for these rare cases (where passivity is preserved upon the addition of physical damping leaving $u_{es}$ unchanged), we will investigate the conditions for finding a new $u_{es}$ for which $\Sigma_1$ can be passivated for a given storage function $H_d$.

3.1 Passivation by interconnection assignment

Although this will be relaxed in subsequent sections, our first result applies for systems where $G$ is constant and has the following form (possibly through variable transformation)

$$G = \begin{bmatrix} 0_{(n-m)\times m} \\ I_m \end{bmatrix}$$

In these cases we define the left annihilator of rank $n - m$ as

$$G^\perp = [I_{n-m} 0_{(n-m)\times m}]$$

Assuming the existence of a control law $u = u_{es} + v$ that transforms $\Sigma_3$ (undamped) into $\Sigma_4$, the latter being passive with respect to $(H_d, v, G^\top \nabla_y H_d)^\top$, the following proposition establishes the condition for the existence of a state feedback $u_{es}' \neq u_{es}$ that transforms the damped system

$^1$ In the sequel we will denote that a system is passive with respect to the triplet $(H, u, y)$ if it is passive with storage function $H$, input $v$ and passive output $y$. 

The matrix presented in Eq. (11) cancels the non actuated terms of $RM^{-1}M_d$ outside the $(n-m)$-order upper left block and removes the skew symmetric part of the latter \(^2\). Hence the $p$-linearly dependent equation is solved with

$$R_2 = \begin{bmatrix} \psi_{n-m}(RM^{-1}M_d) & 0 \\ 0 & 0 \end{bmatrix}$$

Then, taking the matrices $M_d$, $J_{21}$ and the function $V_d$ from the solution of the undamped problem, we obtain a control law

$$u^d = (G^\top G)^{-1}G^\top \{ \nabla_q H + R\nabla_p H - M_dM^{-1}\nabla_q H_d \} + (J_2 - R_2)M_d^{-1}p + v$$

such that along the closed loop trajectories

$$\dot{H}_d = (\frac{\partial H_d}{\partial p})^\top \psi_{n-m}(RM^{-1}M_d) \frac{\partial H_d}{\partial p} + v^\top y \leq v^\top y$$

$\forall (q,p) \in X \times IR^n$, provided that the dissipation condition, holds in $X$. This completes the proof. \(\Box\)

This proposition is instrumental as it provides a criterion to check if physical damping can be an obstacle to achieving passivity in closed loop, and in the positive cases it provides a simple method to construct the passivating control law. For stabilizing the passivated system it is sufficient to add a damping term of the form

$$v = -K_v(q,p)y$$

with $K_v \geq 0$. The procedure illustrated in Proposition 3 for obtaining $u^d$ will be called passivation by interconnection assignment, because it exploits the interconnection matrix $J_{20}$ to cancel of the elements of $RM^{-1}M_d$ outside the critical block $\psi_{n-m}(RM^{-1}M_d)$. This straightforward procedure has a main drawback; it requires the exact knowledge of some elements of matrix $R$, which are friction parameters normally nonconstant and hard to identify experimentally. It is an exact cancellation of friction terms.

Another issue is the possible dependence of $R$ on the momenta $p$. Unless this dependence is strictly linear in $p$, which is a small uninteresting set, the matrix $R$ would still enter the so called $p$-linearly dependent equation, and $J_{20}$ and $R_2$ could freely assume any type of nonlinear dependence on $p$, leaving the main result unchanged except the necessity part in the aforementioned cases.

### 3.2 Passivation by damping injection

An alternative approach that significantly relaxes the parameter identification requirements is the passivation by...
The technique is robust in the sense that any positive matrix $G$ must be satisfied by the interconnection matrix. It assumes the existence of a suitable damping injection method. This technique increases the energy shaping control law used to passivate the system $\Sigma$ into the passive system $\Sigma'$ by computing $\dot{H}_d = -\left( \frac{\partial H_d}{\partial p} \right)^{\top} (C + D) \frac{\partial H_d}{\partial p} + v^{\top} y$ (13) that will be passive if and only if $(C + D) \geq 0$ in $\mathcal{X} \times \mathbb{R}^n$. This new approach aims at passivating the damped system $\Sigma_1$ by simply feeding back the output of $\Sigma_4$, namely $y = G^\top M_d^{-1} p$ and leaving unchanged the interconnection matrix and the energy shaping control law used to passivate the undamped model $u_{es}$. If such a matrix $\bar{R}$ is found, the technique is robust in the sense that any positive matrix $R' > \bar{R}$ would do the job, and hence no exact cancellation of damping terms is needed.

Again, the existence of such a passivating control by damping injection $u_{di}$ is subject to the dissipation condition.

The following proposition applies also in the cases where $G$ is $q$-dependent and it is not integrable, i.e. it can not be transformed into a constant matrix through feedback and variable change. The following proposition of this section extends the passivation by interconnection assignment to the case where $G = G(q)$ with constant rank. The idea is based in Lemma 12.31 of (Nijmeijer and Van der Schaft, 1990).

**Proposition 4.** (Passivation by damping injection) Assume there is an IDA-PBC control law $u = u_{es} + v$ that transforms system $\Sigma_3$ with $G = G(q)$ into a passive system $\Sigma_1$ with respect to $\{v, H_d, G^\top \nabla_p H_d\}$. Then, there exists a passivating output feedback $u_{di} = -R(q)y$ such that $u = u_{es} + u_{di} + v$ transforms the damped system $\Sigma_1$ into a passive system $\Sigma_2$ with $R_d > 0$ if and only if

$$A \overset{\Delta}{=} \text{symm}[G^\top(RM_d^{-1} M_d)(G^\top)^\top] > 0 \quad \forall q \in \mathcal{X}$$

(14)

Furthermore, $R^*$ can be taken diagonal.

**Proof.** (Necessity). Assume that for some $q$ we have $A(q) \leq 0$, that is, there is a nonzero vector $x$ such that $x^\top Ax \leq 0$. Defining the vector $z = (G^\top)^\top x$ and using definitions (12) we have

$$z^\top R_2 z = z^\top (C + D) z = x^\top G^\top C (G^\top)^\top x = x^\top Ax \leq 0,$$

thus $R_2$ cannot be positive definite.

To prove the sufficiency direction we will assume that $A(q) > 0 \forall q \in \mathcal{X}$. Let $V$ be a an $m \times n$ matrix whose columns span the orthogonal complement of $C(\ker G)$. First we prove that the $n \times n$ matrix $[V(G^\top)^\top]$ is nonsingular. Let $V \alpha + (G^\top)^\top \beta = 0$, with $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^{n-m}$. Then $0 = G^\top C(V \alpha + (G^\top)^\top \beta) = G^\top C(G^\top)^\top \beta = A \beta$

Since $A$ is assumed to be positive definite, this implies that $\beta = 0$ and $\alpha = 0$. Then we observe that

$$[V(G^\top)^\top]^\top (C + D) [V(G^\top)^\top] =
\begin{bmatrix}
V^\top CV + V^\top DV & 0 \\
0 & G^\top C(G^\top)^\top
\end{bmatrix}$$

Since

$$\text{rank } V^\top DV = \text{rank } [V(G^\top)^\top]^\top D [V(G^\top)^\top] = \text{rank } D$$

we conclude that as $D = G^\top R^*$, $R_2 = C + D$ can be made positive definite by choosing an appropriate $\hat{R} = R^*$ (if necessary diagonal). This would give a strongly dissipative closed loop system.

### 3.3 Local exponential stability

From Proposition 4 it is clear that if the dissipation condition holds strictly, the system can be made strongly dissipative by damping injection. But for this condition to hold, it
is necessary that det(\(R\)) \(\neq 0\). Hence strong dissipation is a property exclusive to physically damped systems. This feature is very convenient for stability analysis, because as will be shown, strongly dissipative systems are locally exponentially stable (LES). To illustrate this point we will analyze the Jacobian linearization close to the origin, namely

\[
\begin{bmatrix}
\dot{z}_q \\
\dot{z}_p
\end{bmatrix} =
\begin{bmatrix}
0 & M^{-1} \\
-M_d M^{-1} \frac{\partial^2 V_q}{\partial q^2} (J_{\beta_2} - R_d) M_d^{-1}
\end{bmatrix}
\begin{bmatrix}
z_q \\
z_p
\end{bmatrix}
\]

Asymptotic stability of this system will be investigated by defining the positive Lyapunov function

\[
V = \frac{1}{2} z^T Q z, \quad Q \equiv \frac{\partial^2 H_d}{\partial q^2}(0) = \begin{bmatrix} \frac{\partial^2 V_d}{\partial q^2}(0) & 0 \\ 0 & M_d^{-1}(0) \end{bmatrix}
\]

Clearly, \(Q > 0\) in a well designed controller. The time derivative of \(V\) is

\[
\dot{V} = z^T Q \dot{z} = z_p^T M_d^{-1}(0) [J_2(0) - R_d(0)] M_d^{-1}(0) z_p
\]

As we have built a positive definite \(R_d\), the linearized system will converge asymptotically to the largest invariant set where \(z_p \equiv 0\). This set is such that

\[
\dot{z}_p = 0 \Rightarrow -M_d M^{-1} \frac{\partial^2 V_q}{\partial q^2}(0) z_q = 0 \Rightarrow z_q = 0
\]

hence linear asymptotic stability is a fact and local exponential stability is the corollary.

4. EXAMPLE: BALL ON BEAM

This system has been studied in the IDA-PBC framework (see (Ortega et al., 2002)), to obtain a semiglobal stabilizing control law for zero initial velocities. In that paper, physical dissipation has been neglected. As expected from the previous discussion, the closed loop dissipation matrix is not full rank, a situation leading to cumbersome stability proofs and not ensuring local exponential stability.

4.1 System model

The commonly used physical model, under some time and constant scaling (Gordillo et al., 2002), the Euler Lagrange equations become

\[
\ddot{q}_1 + g \sin(q_2) - q_1 \ddot{q}_2 + \beta_1(q,p) q_1 = 0
\]

\[
(L^2 + q_1^2) \ddot{q}_2 + 2q_1 \dot{q}_1 \dot{q}_2 + g q_1 \cos(q_2) + \beta_2(q,p) q_2 = u,
\]

where \(q_1\) is the position of the ball on the beam and \(q_2\) is the angle of the bar, with the origin at the horizontal position. Here we have introduced the positive damping functions \(\beta_1\) and \(\beta_2\) as suggested in (Reddy et al., 2004) but here we also admit the possibility of some dependence on the state.

4.2 Stability of the standard IDA-PBC controller

In (Ortega et al., 2002) an IDA-PBC control law was developed for a damping-free model (i.e setting \(\beta(q,p) = 0\) in (15)), which is not included here for the sake of brevity. As expected from the preceding arguments, the closed loop dissipation matrix is not full rank,

\[
R_d = \begin{bmatrix} 0 & 0 \\ 0 & -k_v \end{bmatrix}
\]

and the asymptotic stability analysis is nontrivial, as when the derivative of the closed loop Hamiltonian

\[
\dot{H}_d = -k_v \left( \frac{p_1 \sqrt{L^2 + q_1^2} - p_2 \sqrt{2}}{(L^2 + q_1^2)^{3/2}} \right)^2,
\]

reaches zero, \(p\) need not be zero.

4.3 Physical damping and nonlinear damping injection

The inclusion of the \(\beta(q,p)\) in the open loop model (15) can induce instability in the IDA-PBC controlled ball on beam system, as was observed by (Reddy et al., 2004) through the linearization at the origin. When trying to understand and solve this problem, one can check that the dissipation condition is trivially satisfied globally. Hence it is always possible to inject enough damping to overcome this difficulty, and this should be done carefully. Moreover, it can be done globally. As physical damping was not considered in (Ortega et al., 2002), the stability could be compromised for some values of \(\beta\). Here we will design the damping injection terms to get a globally positive definite closed loop dissipation matrix. As this is a small dimension problem, the procedure for passivation is simple, even with uncertainties in the dissipation terms. However the novelty with respect to the Instead of the linear output feedback used in (Ortega et al., 2002), a nonlinear damping control law of the form

\[
u_{di} = -k_v(q,p)y = -k_v(q,p)G^T \nabla_p H_d
\]

will be needed for the system to be globally strongly dissipative. Actually this happens when the closed loop dissipation matrix is globally positive, that is,

\[
k_v > \frac{1}{2 \sqrt{2}} \frac{-6 \beta_1 (L^2 + q_1^2) \beta_2 + \beta_1^2 (L^2 + q_1^2)^2 + \beta_2^2}{\sqrt{L^2 + q_1^2}}
\]

This can be satisfied with a constant \(k_v\) on any compact set. However, for \(q \in \mathbb{R}^n\) there is no constant output feedback satisfying this equation, thus we recourse to a state dependent form of \(k_v\) like

\[
u_{di} = -\frac{\beta_1^2 (L^2 + q_1^2)^2 + \beta_2^2}{2 \sqrt{L^2 + q_1^2}} \left( \frac{p_1 \sqrt{L^2 + q_1^2} - p_2 \sqrt{2}}{(L^2 + q_1^2)^{3/2}} \right)
\]

where

\[
\beta_1 > \max_{(q,p)}(\beta_1(q,p)) \quad \beta_2 > \max_{(q,p)}(\beta_2(q,p))
\]

are some estimated upper bounds on the friction parameters. With this controller parameters, the closed loop system is locally exponentially stable and it in virtue of the strong dissipation it can be easily proved that the trajectories converge to the set

\[
\{q \in \mathbb{R}^n | \nabla V_d(q) = 0\} \cap \{p = 0\}
\]

which is a countable set of isolated points of the form

\[
\bar{q} = (L \sinh(\sqrt{2} i \pi), i \pi), \quad i \in \mathbb{N}
\]
including the origin and other points outside \( \{ q_2 \in (\pi, \pi) \} \). This results simplifies the sometimes difficult stability proofs of underactuated control.

5. SIMULATIONS

System (15) will be simulated with the energy shaping control law from (Ortega et al., 2002) and the two possible damping injection terms discussed in Section 4.3. First, we will use a constant linear feedback as proposed in (Ortega et al., 2002) (setting \( k_v > 0 \) constant) and secondly the nonlinear output feedback (16). While for sufficiently large \( k_v \) both controllers will work fine locally, for initial conditions further away from the origin the linear output feedback will be insufficient to keep \( H_d \) always decreasing, whereas (16) ensures global dissipation.

Figure 2 depicts the simulation results of the ball and beam under different dissipation conditions. The three graphs in the upper row show the trajectories of \( q_1 \) and \( q_2 \) vs. time. The lower row shows the time dependence of the closed loop Hamiltonian function \( H_d \) corresponding to each trajectory in the above graph. Three different conditions have been simulated. The first case, (graphs (a1) and (a2)), illustrates the PBC controller with \( k_v \) constant acting on a damping-free model, as in (Ortega et al., 2002). For any \( k_v > 0 \), the semidefinite dissipation matrix is sufficient to ensure stability and no further considerations must be done. The second simulation, (b1) and (b2) shows how the performance of the constant \( k_v \) controller is downgraded when physical damping is introduced in the model an not considered for design. Figure (b2) has been zoomed in to stress out that the closed loop energy is not monotonic: stability can be compromised. Graphs (c1) and (c2) show the closed loop behavior of the physically damped system when the nonlinear damping term (16) is added to the controller. This controller recovers a monotonic Lyapunov function for every initial state, even without exact knowledge of the \( \beta \) parameters.

6. CONCLUSIONS

In this paper the IDA-PBC control technique for underactuated mechanical systems has been extended to incorporate an important phenomenon that has been neglected in previous related works: open loop damping. Given a solution of the IDA-PBC matching equations for the undamped model, this paper discusses the possibility of control redesign in order to maintain passivity when friction appears.

REFERENCES


