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Department of Applied Mathematics, University of Twente, Enschede, The Netherlands

A note on the lower bound for online strip packing

W. Kern and J.J. Paulus

Abstract

This note presents a lower bound of $3/2+\sqrt{33}/6 \approx 2.457$ on the competitive ratio for online strip packing. The instance construction we use to obtain the lower bound was first coined by Brown, Baker and Katseff [2]. Recently this instance construction is used to improve the lower bound in computer aided proofs. We derive the best possible lower bound that can be obtained with this instance construction.

1 Introduction

In the two-dimensional strip packing problem a number of rectangles have to be packed without rotation or overlap into a strip such that the height of the strip used is minimum. The width of the rectangles is bounded by 1 and the strip has width 1 and infinite height. Baker, Coffman and Rivest [1] show that this problem is NP-hard.

We study the online version of this packing problem. In the online version the rectangles are given to the online algorithm one by one from a list, and the next rectangle is given as soon as the current rectangle is irrevocably placed into the strip. To evaluate the performance of an online algorithm we employ competitive analysis. For a list of rectangles L, the height of a strip used by online algorithm A and by the optimal solution is denoted by $A(L)$ and $OPT(L)$, respectively. The optimal solution is not restricted in any way by the ordering of the rectangles in the list. Competitive analysis measures the absolute worst-case performance of online algorithm A by its competitive ratio $\sup_L\{A(L)/OPT(L)\}$.

In the early 80's, a lower bound of 2 on the competitive ratio is given by Brown, Baker and Katseff [2]. More recently, improved lower bounds have successively been obtained by Johannes [5] and Hurink and Paulus [3], a lower bound of 2.25 and 2.43, respectively. Both results are obtained in the setting of online parallel job scheduling, a closely related problem, and apply directly to the online strip packing problem. These lower bounds are obtained by using the aid of a computer program; the first uses an enumerative process and the second an ILP-solver. It is interesting to note that all lower bounds for online strip packing are based on the same instance construction. The next section describes this construction. It was shown by Hurink and Paulus [3] that this construction cannot lead to a lower bound higher than

2.5. This note closes the gap between 2.43 and 2.5, by proving a lower bound of $3/2 + \sqrt{33}/6 \approx 2.457$ on the competitive ratio, and showing that this is the best possible bound that can be obtained by this instance construction.

Regarding the upper bound on the competitive ratio for online strip packing, recent advances have been made by Ye, Han and Zhang [6] and Hurink and Paulus [4]. Independently they present an online algorithm with competitive ratio $7/2 + \sqrt{10} \approx 6.6623$, that is a modification of the well known shelf algorithm. We refer to these two papers for a more extensive overview of the literature.

2 The instance construction

In this section we formalize the instance construction used to obtain the lower bound. Additionally, we present an online algorithm for packing the rectangles to show that no lower bound larger that $3/2+\sqrt{33}/6$ can be obtained by this instance construction. For convenience let throughout this note $\rho = 3/2 + \sqrt{33}/6$.

We define L_n as the list of rectangles $(p_0, q_1, p_1, q_2, p_2, \ldots, q_n, p_n)$, where p_i denotes a rectangle of height p_i and width no more than $1/(n+1)$, and q_i denotes a rectangle of height q_i and width 1. The rectangle heights are defined as

$$\begin{aligned}
p_0 &= 1 \ , \\
p_i &= \beta_{i-1} p_{i-1} + p_{i-1} + \alpha_i p_i + \epsilon \quad \forall i \geq 2 \ , \\
q_1 &= \beta_0 p_0 + \epsilon \ , \\
q_i &= \max\{\alpha_{i-1} p_{i-1}, q_{i-1}, \beta_{i-1} p_{i-1}\} + \epsilon \quad \forall i \geq 2 \ ,
\end{aligned}$$

where $\alpha_i p_i$ and $\beta_i p_i$ are the distances the online algorithm has placed between earlier rectangles, and ϵ is a small positive value. The value $\alpha_i p_i$ denotes the vertical distance between rectangles p_{i-1} and q_i, and the value $\beta_i p_i$ denotes the vertical distance between q_i and p_i. This is illustrated in Figure 1. The values α_i and β_i completely characterize the behavior of the online algorithm when processing L_n.

By definition of the rectangles' heights and widths, an online algorithm can only pack the rectangles one above the other in the same order as the rectangles appear in the list L_n. An optimal packing is obtained by first packing the rectangles q_i on top of each other and then pack all p_i next to each other on top of the q-rectangles. The sole purpose of the positive term ϵ is to ensure this structure on any online packing. From now on we assume that ϵ is small enough to be omitted from the analysis.

Before proving the lower bound of ρ on the competitive ratio in the next section, we show the limitation of using the list L_n.

Theorem 1. *With the list L_n, no lower bound on the competitive ratio larger than $\rho = \frac{3}{2} + \frac{\sqrt{33}}{6}$ can be obtained for online strip packing.*

Proof. Consider the online algorithm A that chooses $\beta_0 = \rho - 1$, $\alpha_2 = 1/(\rho - 1)$, and all other gaps equal to 0. This algorithm is ρ-competitive when presented with L_n:

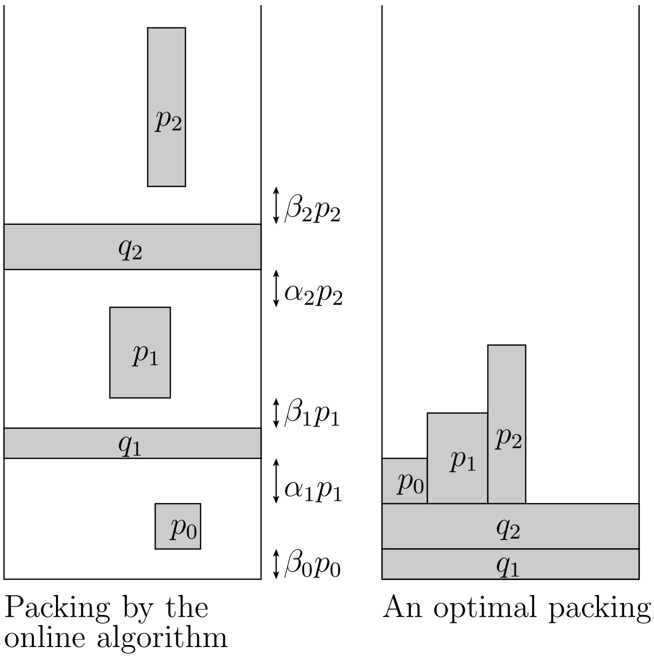

Figure 1: Online and optimal packing of L_2.

- After packing rectangle p_0 we have $A(L_0) = \rho$ and $OPT(L_0) = 1$. Thus, the competitive ratio is exactly ρ at this point.
- By packing rectangle q_1 the online and optimal packing increase by the same amount. Thus the competitive ratio decreases.
- After packing p_1 we have $A(L_1)/OPT(L_1) = (3\rho - 1)/(2\rho - 1) < \rho$.
- After packing q_2 we have $A(L_1 q_2)/OPT(L_1 q_2) = (4\rho - 2 + \alpha_2 p_2)/(3\rho - 2)$, with $p_2 = \rho + \alpha_2 p_2$. By choice of α_2, we have $\alpha_2 p_2 = 3\rho - 2$ and $A(L_1 q_2)/OPT(L_1 q_2) = (7\rho - 4)/(3\rho - 2) = \rho$. Again the competitive ratio is exactly ρ at this point. (This last equality motivates the value of ρ).
- After packing p_2 we have $A(L_2)/OPT(L_2) = (11\rho - 6)/(6\rho - 4) < \rho$.
- For $i \geq 3$ there are no more gaps introduced by online algorithm A. By packing q_i the online and optimal packing increase by the same amount and, thus, ρ-competitiveness is not violated. By choice of α_2 we have $q_i = p_i/(\rho - 1)$. This implies $OPT(L_{i+1}) = OPT(L_i) + q_{i+1}$ and $A(L_{i+1}) = A(L_i) + q_{i+1} + p_{i+1} = A(L_i) + \rho q_{i+1}$. The height used in the online packing grows exactly ρ times as fast as the optimal packing.

So, online algorithm A is ρ-competitive for the list rectangles L_n. □

3 Lower bound on the competitive ratio

In this section we prove a lower bound of $\rho = 3/2 + \sqrt{33}/6$ on the competitive ratio for online strip packing. The outline of the proof is as follows.

To prove that no online algorithm can have a competitive ratio smaller than ρ, we assume that there exists a $(\rho - \delta)$-competitive online algorithm A (with $\delta > 0$). We present this algorithm with the list L_n, with n arbitrarily large. To obtain a contradiction we define a potential function Φ_i on the state of the online packing after packing rectangle p_i. We argue that this potential function is both bounded from below and that it decreases to $-\infty$, giving us the required contradiction.

For convenience let $\tilde{\rho} = \rho - \delta$. After packing the rectangle p_i, we measure with γ_i how much online algorithm A improves upon the $\tilde{\rho}$-competitiveness bound: We define γ_i through

$$A(L_i) + \gamma_i p_i = \tilde{\rho} OPT(L_i) \ .$$

The potential function Φ_i is defined (after packing rectangle p_i) by

$$\Phi_i := \frac{\gamma_i + \beta_i - (\tilde{\rho} - 2)\alpha_i}{1 - \alpha_i} \ .$$

The values of α_i and β_i are nonnegative, $1 - \alpha_i$ is positive by definition of p_i and γ_i is nonnegative by the $\tilde{\rho}$-competitiveness of online algorithm A. In Lemma 6 we show among other things that $\alpha_i < 1/(\rho - 1)$. As a consequence $\Phi_i > -1$ for all i. In Lemma 7 we show that $\Phi_{i+1} \leq \Phi_i - \delta$ for all i. With n large enough, these two results are contradicting. This proves the main result of this note:

Theorem 2. *No online algorithm for online strip packing is* $(3/2 + \sqrt{33}/6 - \delta)$*-competitive, with* $\delta > 0$*.* □

The remainder of this note is concerned with the proofs of the afore mentioned lemmata.

Proof of the lemmata

Before showing that $\alpha_i < 1/(\rho - 1)$ and $\Phi_{i+1} \leq \Phi_i - \delta$, we derive in Lemma 1 to Lemma 5 some basic properties of the potential function Φ_i. In the following δ is assumed to be a fairly small positive value. For relatively large δ we already know from previous work [2, 3, 5] that no $(\rho - \delta)$-competitive algorithm exists.

Lemma 1. *The potential* Φ_i *is invariant under shifting* p_i*.*

Proof. Shifting rectangle p_i up or down does not affect $OPT(L_i)$, α_i or its own length. However, it does change β_i and γ_i by the same amount but with opposite sign, i.e. the sum $\beta_i + \gamma_i$ is constant. Hence, Φ_i is invariant under shifting p_i. □

Lemma 2. $(\beta_i + 1)\, p_i = (1 - \alpha_{i+1})\, p_{i+1} \ .$

Proof. By definition of the list L_n. ☐

Lemma 3.

$$\Phi_{i+1} = \frac{\gamma_i + (\tilde{\rho}-1)\beta_i - 1 + (\tilde{\rho}-1)q_{i+1}/p_i}{1+\beta_i} \ .$$

Proof. By Lemma 1 we can shift rectangle p_{i+1} down without affecting Φ_{i+1}, i.e. $\beta_{i+1} = 0$. Then

$$\begin{aligned}
(1-\alpha_{i+1})p_{i+1}\Phi_{i+1} &= (\gamma_{i+1} + \beta_{i+1} - (\tilde{\rho}-2)\alpha_{i+1})\, p_{i+1} \\
&= (\gamma_{i+1} - (\tilde{\rho}-2)\alpha_{i+1})\, p_{i+1} \\
&= \tilde{\rho} OPT(L_{i+1}) - A(L_{i+1}) - (\tilde{\rho}-2)\alpha_{i+1}p_{i+1} \\
&= \tilde{\rho}\, (OPT(L_i) + q_{i+1} + \beta_i p_i + \alpha_{i+1}p_{i+1}) \\
&\quad - (A(L_i) + \alpha_{i+1}p_{i+1} + q_{i+1} + p_{i+1}) - (\tilde{\rho}-2)\alpha_{i+1}p_{i+1} \\
&= \gamma_i p_i + \tilde{\rho}\beta_i p_i - (1-\alpha_{i+1})p_{i+1} + (\tilde{\rho}-1)q_{i+1} \\
&= (\gamma_i + (\tilde{\rho}-1)\,\beta_i - 1)\, p_i + (\tilde{\rho}-1)\, q_{i+1} \ .
\end{aligned}$$

The last equality is due to Lemma 2. By Lemma 2 we can divide the left hand side by $(1-\alpha_{i+1})p_{i+1}$ and the right hand side by $(1+\beta_i)p_i$ to obtain the result. ☐

Lemma 4. *If $q_{i+1} = \max\{\alpha_i p_i, q_i\}$, then we may assume w.l.o.g. that $\beta_i = 0$.*

Proof. Shifting rectangle p_i down decreases the distance $\beta_i p_i$ and increases $\alpha_{i+1}p_{i+1}$. However, when we keep all other distances equal it does not affect p_j with $j > i$. Due to the increase in $\alpha_{i+1}p_{i+1}$ some q_j with $j > i$ may increase, but this is only in favor of the online algorithm since the optimal value increases by the exact same amount. ☐

Lemma 5. *If $\Phi_i \leq \tilde{\rho} - 1$ then $\gamma_i + \beta_i + \alpha_i \leq \tilde{\rho} - 1$.*

Proof.

$$\begin{aligned}
\Phi_i \leq \tilde{\rho} - 1 \quad &\Rightarrow \quad \gamma_i + \beta_i - (\tilde{\rho}-2)\alpha_i \leq (\tilde{\rho}-1)(1-\alpha_i) \\
&\Rightarrow \quad \gamma_i + \beta_i + \alpha_i \leq \tilde{\rho} - 1 \ .
\end{aligned}$$

☐

Lemma 6. *For $i \geq 0$, $\Phi_i \leq \tilde{\rho} - 1$, and for $i \geq 1$, $\alpha_i \leq 1/(\rho-1) - \delta/4$ and $q_i/p_i \leq 1/(\rho-1) - \delta/4$.*

Proof. We prove the three inequalities simultaneously by induction. The lemma holds for $i = 0$ since $\gamma_0 + \beta_0 \leq \tilde{\rho} - 1$ and thus $\Phi_0 = \tilde{\rho} - 1$. We assume the lemma holds up to i, and then show it for $i + 1$. We make a case distinction on the way the height of rectangle q_{i+1} is determined.

<u>Case 1:</u> $q_{i+1} = \alpha_i p_i$.

By Lemma 4 we can assume $\beta_i = 0$. Thus

$$\frac{q_{i+1}}{p_{i+1}} = \frac{\alpha_i p_i}{p_i + \alpha_{i+1} p_{i+1}} \leq \alpha_i \leq \frac{1}{\rho - 1} - \frac{\delta}{4} \ .$$

The online algorithm A is by assumption $\tilde{\rho}$-competitive after packing rectangle q_{i+1}, which means that the distance between rectangles q_{i+1} and p_i is not too large, i.e. $\alpha_{i+1} p_{i+1} \leq \gamma_i p_i + (\tilde{\rho} - 1) q_{i+1} = \gamma_i p_i + (\tilde{\rho} - 1)\alpha_i p_i$. Together with Lemma 5 this gives

$$\begin{aligned} \alpha_{i+1} &= \frac{\alpha_{i+1} p_{i+1}}{p_i + \alpha_{i+1} p_{i+1}} \leq \frac{\gamma_i + (\tilde{\rho} - 1)\alpha_i}{1 + \gamma_i + (\tilde{\rho} - 1)\alpha_i} \\ &\leq \frac{(\tilde{\rho} - 1)^2}{1 + (\tilde{\rho} - 1)^2} < \frac{1}{\rho - 1} - \frac{\delta}{4} \ . \end{aligned}$$

By Lemma 3, the induction assumption and Lemma 5 we get

$$\Phi_{i+1} = \gamma_i - 1 + (\rho - 1)\alpha_i < \gamma_i \leq \tilde{\rho} - 1 \ .$$

Case 2: $q_{i+1} = \beta_i p_i$.

By Lemma 5 we have $\beta_i \leq \rho - \delta - 1$ and thus

$$\frac{q_{i+1}}{p_{i+1}} = \frac{\beta_i p_i}{(1 + \beta_i) p_i + \alpha_{i+1} p_{i+1}} \leq \frac{\beta_i}{1 + \beta_i} \leq \frac{\tilde{\rho} - 1}{\tilde{\rho}} \ . \tag{1}$$

Note that $(\tilde{\rho} - 1)/\tilde{\rho} < 1/(\rho - 1) - \delta/4$.

The online algorithm A is by assumption $\tilde{\rho}$-competitive after packing rectangle q_{i+1}, which means that the distance between rectangles q_{i+1} and p_i is not too large, i.e. $\alpha_{i+1} p_{i+1} \leq \gamma_i p_i + (\tilde{\rho} - 1) q_{i+1} = \gamma_i p_i + (\tilde{\rho} - 1)\beta_i p_i$. This, together with $\beta_i + \gamma_i \leq \tilde{\rho} - 1$ (by Lemma 5) gives

$$\begin{aligned} \alpha_{i+1} &= \frac{\alpha_{i+1} p_{i+1}}{(1 + \beta_i) p_i + \alpha_{i+1} p_{i+1}} \leq \frac{\gamma_i + (\tilde{\rho} - 1)\beta_i}{1 + \beta_i + \gamma_i + (\tilde{\rho} - 1)\beta_i} \\ &= \frac{\gamma_i + (\tilde{\rho} - 1)\beta_i}{1 + \gamma_i + \tilde{\rho}\beta_i} \leq \frac{(\tilde{\rho} - 1)^2}{1 + (\tilde{\rho})(\tilde{\rho} - 1)} < \frac{1}{\rho - 1} - \frac{\delta}{4} \ , \end{aligned}$$

(The second to last inequality is found by maximizing the left hand side, i.e. $\gamma_i = 0$ and $\beta_i = \tilde{\rho} - 1$.) and

$$\Phi_{i+1} = \frac{\gamma_i + 2(\tilde{\rho} - 1)\beta_i - 1}{1 + \beta_i} \leq \frac{2(\tilde{\rho} - 1)^2 - 1}{\tilde{\rho}} \ . \tag{2}$$

Note that $(2(\tilde{\rho} - 1)^2 - 1)/\tilde{\rho} < \tilde{\rho} - 1$.

Case 3: $q_{i+1} = q_i$.

By induction we get

$$\frac{q_{i+1}}{p_{i+1}} = \frac{q_i}{p_{i+1}} \leq \frac{q_i}{p_i} \leq \frac{1}{\rho - 1} - \frac{\delta}{4} \ .$$

By Lemma 4 we can assume $\beta_i = 0$, and thus

$$\begin{aligned} \Phi_{i+1} &= \gamma_i - 1 + (\tilde{\rho} - 1)\frac{q_{i+1}}{p_i} \\ &\leq \gamma_i - 1 + (\tilde{\rho} - 1)(\frac{1}{\rho - 1} - \frac{\delta}{2}) \leq \gamma_i \leq \tilde{\rho} - 1 \ . \end{aligned}$$

To argue that $\alpha_{i+1} \leq 1/(\rho-1) - \delta/4$ we shift the rectangles $(q_i, p_i, q_{i+1}, p_{i+1}, \dots)$ all by the same distance down until either $\gamma_i = 0$ or $\alpha_i = 0$. By shifting $(q_i, p_i, q_{i+1}, p_{i+1}, \dots)$ down only the length of p_i decreases, therefore γ_i can become 0. Since the optimal solution is not affected by this shift, the online algorithm is still $\tilde{\rho}$-competitive.

If $\gamma_i = 0$, then $\alpha_{i+1}p_{i+1} \leq (\tilde{\rho} - 1)q_{i+1} = (\tilde{\rho} - 1)q_i \leq (\tilde{\rho} - 1)p_i/(\rho - 1) \leq p_i$, and thus $\alpha_{i+1} \leq 1/(\rho - 1) - \delta/4$.

If $\alpha_i = 0$, then rectangles p_{i-1}, q_i, p_i are concatenated. To show that $\alpha_{i+1} \leq 1/(\rho - 1) - \delta/4$ also holds for this case, we make three more case distinctions.

<u>Case 3a:</u> $q_{i+1} = q_i = \alpha_{i-1}p_{i-1}$.

By Lemma 4 we can assume $\beta_{i-1} = 0$, implying that $p_i = p_{i-1}$. First note that $\gamma_i p_i = \gamma_{i-1}p_{i-1} + (\tilde{\rho} - 1)q_i - p_i \leq \gamma_{i-1}p_{i-1}$, and $\gamma_{i-1} + \beta_{i-1} + \alpha_{i-1} \leq \tilde{\rho} - 1$. Thus

$$\begin{aligned} \alpha_{i+1}p_{i+1} &\leq (\tilde{\rho} - 1)q_{i+1} + \gamma_i p_i \\ &\leq (\tilde{\rho} - 1)\alpha_{i-1}p_{i-1} + \gamma_{i-1}p_{i-1} \\ &\leq (\tilde{\rho} - 1)\alpha_{i-1}p_{i-1} + (\tilde{\rho} - 1 - \alpha_{i-1})p_{i-1} \\ &\leq (\tilde{\rho} - 1)p_{i-1} + (\tilde{\rho} - 2)\alpha_{i-1}p_{i-1} \ , \end{aligned}$$

and therefore

$$\alpha_{i+1} = \frac{\alpha_{i+1}p_{i+1}}{p_i + \alpha_{i+1}p_{i+1}} \leq \frac{\tilde{\rho} - 1 + (\tilde{\rho} - 2)\alpha_{i-1}}{\tilde{\rho} + (\tilde{\rho} - 2)\alpha_{i-1}} < \frac{1}{\rho - 1} - \frac{\delta}{4} \ .$$

(The last inequality is found by maximizing the left hand side, i.e. $\alpha_{i-1} = 1/(\rho - 1) - \delta/4$.)

<u>Case 3b:</u> $q_{i+1} = q_i = \beta_{i-1}p_{i-1}$.

$$\begin{aligned} \alpha_{i+1}p_{i+1} &\leq (\tilde{\rho} - 1)q_{i+1} + \gamma_i p_i = (\tilde{\rho} - 1)q_i + \gamma_i p_i \\ &\leq_{\text{by (1)}} \left((\tilde{\rho} - 1)^2/\tilde{\rho} + \gamma_i\right) p_i \ . \end{aligned}$$

Since after shifting rectangles (q_i, p_i) down until $\alpha_i = 0$, we may conclude from (2) that $\Phi_i = \gamma_i \leq (2(\tilde{\rho} - 1)^2 - 1)/\tilde{\rho}$. Thus

$$\alpha_{i+1} = \frac{\alpha_{i+1}p_{i+1}}{p_i + \alpha_{i+1}p_{i+1}} \leq \frac{(\tilde{\rho} - 1)^2/\tilde{\rho} + \gamma_i}{1 + (\tilde{\rho} - 1)^2/\tilde{\rho} + \gamma_i} \leq \frac{1}{\rho - 1} - \frac{\delta}{4} \ .$$

(The last inequality is found by maximizing the left hand side, i.e. $\gamma_i \leq (2(\tilde{\rho} - 1)^2 - 1)/\tilde{\rho}$)

Case 3c: $q_{i+1} = q_i = q_{i-1}$.

By Lemma 4 we can assume $\beta_{i-1} = 0$. After shifting rectangles (q_i, p_i) down until $\alpha_i = 0$, the rectangles $(q_{i-1}, p_{i-1}, q_i, p_i)$ are concatenated. Because the "competitiveness" of online algorithm A decreases in the sense that $\gamma_i p_i = \gamma_{i-1}p_{i-1} + (\tilde{\rho}-1)q_i - p_i \leq \gamma_{i-1}p_{i-1}$, the maximum possible value of α_{i+1} is smaller than that of α_i. So the claim follows by induction.

□

Lemma 7. $\Phi_{i+1} \leq \Phi_i - \delta$.

Proof. By a number of case distinctions we show that $\Phi_{i+1} \leq \Phi_i - \delta$.

Case 1: $q_{i+1} = \alpha_i p_i$.

By Lemma 4 we can assume $\beta_i = 0$. By Lemma 3 we have

$$\Phi_{i+1} - \Phi_i = \gamma_i - 1 + (\tilde{\rho} - 1)\alpha_i - \frac{\gamma_i - (\tilde{\rho} - 2)\alpha_i}{1 - \alpha_i} \ .$$

The derivative with respect to γ_i of the above is $1 - 1/(1 - \alpha_i) \geq 0$. Hence, $\Phi_{i+1} - \Phi_i$ is large for large γ_i. Choose therefore $\gamma_i = \tilde{\rho} - 1$. For $\delta \in [0, \rho - 2]$ we have

$$\Phi_{i+1} - \Phi_i \leq \tilde{\rho} - 2 + (\tilde{\rho} - 1)\alpha_i - \frac{\tilde{\rho} - 1 - (\tilde{\rho} - 2)\alpha_i}{1 - \alpha_i} \leq -\delta \ .$$

Case 2: $q_{i+1} = \beta_i p_i$.

By Lemma 3 we have

$$\Phi_{i+1} - \Phi_i = \frac{\gamma_i + 2(\tilde{\rho} - 1)\beta_i - 1}{1 + \beta_i} - \frac{\gamma_i + \beta_i - (\tilde{\rho} - 2)\alpha_i}{1 - \alpha_i} \ .$$

The derivative with respect to γ_i of the above is $1/(1 + \beta_i) - 1/(1 - \alpha_i) \leq 0$. The difference $\Phi_{i+1} - \Phi_i$ is decreasing in γ_i, so choose $\gamma_i = 0$. Additionally, we have that $\alpha_i \leq \beta_i$, otherwise we are not in this case. With $\gamma_i = 0$ and under the constraint $\alpha_i \leq \beta_i \leq 1/(\rho - 1) - \delta/4$ we have

$$\Phi_{i+1} - \Phi_i \leq \frac{2(\tilde{\rho} - 1)\beta_i - 1}{1 + \beta_i} - \frac{\beta_i - (\tilde{\rho} - 2)\alpha_i}{1 - \alpha_i} < -\delta \ .$$

Case 3: $q_{i+1} = q_i$.

By Lemma 4 we can assume $\beta_i = 0$. Lemmata 3 and 6 imply

$$\begin{aligned}
\Phi_{i+1} - \Phi_i &= \gamma_i - 1 + (\tilde{\rho} - 1)\frac{q_{i+1}}{p_i} - \frac{\gamma_i - (\tilde{\rho} - 2)\alpha_i}{1 - \alpha_i} \\
&\leq \gamma_i - 1 + (\tilde{\rho} - 1)(\frac{1}{\rho - 1} - \frac{\delta}{2}) - \frac{\gamma_i - (\tilde{\rho} - 2)\alpha_i}{1 - \alpha_i} \ .
\end{aligned}$$

The derivative of the right hand side with respect to γ_i is $1 - 1/(1 - \alpha_i) \geq 0$. So choose γ_i large, i.e. $\gamma_i = \tilde{\rho} - 1$. This results in

$$\Phi_{i+1} - \Phi_i \leq -\delta \ .$$

In each case there is a substantial decrease in the potential function. □

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