

On the Solution, the Critical Exponents and the Transition Equation of the Simple Cubic Three-dimensional Ising Model

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Abstract

In a recent paper Hoede and Zandvliet introduced the concept of gauging on an equation. This enables the simulation of more complex Ising models by the simple quadratic model. The possibility of simulating the simple cubic model was defended by calculating a sequence of approximations to the transition point from a so-called solution function.

In this paper we investigate some aspects of such a simulation, in particular with respect to the solution. It is also shown that taking into account the difference in universality class is possible. It is argued that the critical exponent for the specific heat is 0. The theory then also leads to a choice of the other critical exponents, depending on the validity of the scaling hypothesis of Widom and experimental data. Using the value $\frac{3}{8}$ for the critical exponent for the magnetization we establish a relationship between the transition equations of the 2D and the 3D model.

Key words: Ising model, critical exponents, transition equation

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1 Introduction

We refer to the paper of Hoede and Zandvliet [6] for the basic definitions and the conceptual background of this paper.

The basic idea is that the 3D-Ising model can be simulated by the 2D-Ising model, where in both cases the simplest model is considered. These are the simple cubic lattice respectively the simple quadratic lattice, both with isotropic interactions.

With interaction energy J , Boltzmann constant k_B and temperature T , $H = \frac{J}{k_B T}$ is the variable considered in the 3D-case, whereas $H^* = \frac{J^*}{k_B T}$ is the variable considered for the simulating 2D-model.

The solution function $y=S(x)$ was calculated from the series expansions of the susceptibility in terms of $y=\tanh(H^*)$ respectively $x=\tanh(H)$ by equating the expansions. The solution function reads

$$y = 3(x/2) + 3(x/2)^2 + 3(x/2)^3 + 69(x/2)^4 + 459(x/2)^5 + 411(x/2)^6 + 22791(x/2)^7 + 66213(x/2)^8 + 1154723(x/2)^9 + 3786801(x/2)^{10} + 69728919(x/2)^{11} + 236351403(x/2)^{12} + 4391519223(x/2)^{13} + 14977375443(x/2)^{14} + 290669917619(x/2)^{15} + 1017776595693(x/2)^{16} + 19566870142959(x/2)^{17} + 71084173367165(x/2)^{18} + 1357967026040259(x/2)^{19} + 5017309562668557(x/2)^{20} + 96031088694860083(x/2)^{21}.$$

The transition equation for the 2D-model was conjectured by Kramers and Wannier [7] in 1941 to be

$$\sinh(2H_c^*) = 1,$$

from which follows the critical value H_c^* , for which $y_c = \tanh(H_c^*) = \sqrt{2} - 1$. Solving the equation $S(x_c) = \sqrt{2} - 1$, where $x_c = \tanh(H_c)$, leads to the value

$$H_c = 0.2256$$

for the 3D-model. Taking the terms of $S(x)$ into account step by step gives a sequence that monotonously descends. The best known value for H_c , found by Monte Carlo methods, was given by Blöte *et al.* [3] and is

$$H_c = 0.2216544,$$

where the last digit is uncertain.

The transition equation for the 2D-model reads

$$\sinh(2H_x^*)\sinh(2H_y^*) = 1$$

in the anisotropic case with interaction energies J_x^* and J_y^* and $H_x^* = \frac{J_x^*}{k_B T}$ respectively $H_y^* = \frac{J_y^*}{k_B T}$ and was confirmed by Onsager [12] in 1944 by his well-known solution of the 2D-model. The free energy shows a logarithmic singularity. We will indicate Onsager's solution by $O(H_x^*, H_y^*)$ or just by $O(H^*)$ in case we consider the isotropic lattice.

The idea of simulation in [6] was simply to consider $H^* = \operatorname{arctanh}(S(\tanh(H)))$ and substitute this expression in H for H^* in $O(H^*)$ to obtain $O(H)$ as approximate solution for the 3D-model.

There is a direct objection against this procedure. Consider the situation in which the temperature T tends to 0, then both x and y are tending to 1. However, then $S(x)$ grows beyond the value 1. In fact $S(x)=1$ for $x=0.277053$. The conclusion must be that the simulating solution, obtained in this way, can only be valid in the direct neighborhood of

the critical point.

There is, however, a rather stringent consequence of the fact that the transition points of the 3D-model and that of a simulating 2D-model correspond. Let J^* be the interaction energy for the simulating 2D-model and let J be the interaction energy for the 3D-model. We know that

$$\frac{J^*}{k_B T_{c,2D}} = 0.4406868\dots$$

and

$$\frac{J}{k_B T_{c,3D}} = 0.2216544\dots$$

The simulating 2D-model should have the same transition **temperature**, i.e. we must have $T_{c,2D} = T_{c,3D}$. But this implies

$$\frac{J^*}{0.4406868} \approx \frac{J}{0.2216544}$$

or

$$J^* = f_c \cdot J,$$

where

$$f_c = \frac{0.4406868\dots}{0.2216544\dots} = 1.9881707\dots$$

is a constant, determined by the two critical points H_c^* and H_c . As $f_c = \frac{H_c^*}{H_c}$, the relation between J^* now immediately gives

$$O_{3D}(H) = O_{2D}(f_c \cdot H)$$

as approximate solution for the 3D-model. Note that H_c is not exactly known, as H_c^* is. This, esthetically appealing, result is based on the assumption that simulation is indeed possible and on the findings in [6] on the correspondence of critical points.

The quotient between H^* and H need not be a constant. We rather expect that

$$H^* = f(H) \cdot H,$$

where $f(H)$ is a *simulation strength factor* depending on the temperature. We have $f(H_c) = f_c$ by the argument given above, but there is also a physical argument determining the limit of $f(H)$ for $T \rightarrow 0$, i.e. $H \rightarrow \infty$. At $T=0$ we consider a lattice with only (-)-spins, in which one (+)-spin is present. The energy needed for its existence is $4.2J^* = 8J^*$ in the simulating 2D-situation. In the simulated 3D-situation the energy needed is $6.2J = 12J$. For the simulation these energies should be equal, leading to

$$J^* = \frac{3}{2}J,$$

or

$$H^* = \frac{3}{2}H$$

near $T=0$. Hence $\lim_{H \rightarrow \infty} f(H) = \frac{3}{2}$, a result for which we will give an extra argument later.

Our approach transforms the problem to finding the correct simulation strength factor, that seems to vary slowly from $\frac{3}{2}$ to $f_c \approx 2$, there where the magnetization is non-zero. Our reasoning was the following.

In [6] we proposed an alternative approach that boiled down to coming forward with a potential solution, like we just did, and providing a truth certificate, e.g. by consistence with series developments. We chose the Monte Carlo results of Talapov and Blöte [13] for the magnetization of the 3D-model as results to compare with.

For the magnetization of the 2D-model we have Yang's result [15]

$$M_{2D} = [1 - (\frac{1}{\sinh(2H^*)})^4]^{\frac{1}{8}}.$$

Talapov and Blöte give

$$M_{3D}(t) = t^{0.32694109}(1.6919045 - 0.34357731.t^{0.50842026} - 0.42572366.t),$$

where $t = 1 - 0.2216544\frac{k_B T}{J}$. We quote: "This formula may serve as a very accurate empirical approximation for the spontaneous magnetization of the simple cubic Ising model in the region $0.0005 \leq t \leq 0.26$."

Note that the more familiar form of t is

$$t = 1 - \frac{H_c}{H} = 1 - \frac{T}{T_c} = \frac{T_c - T}{T_c}.$$

In M_{2D} we first considered a constant simulation strength factor f_c . We plotted both expressions for the magnetization. Both are zero for the critical temperature, the simulating model giving $M_{2D}(H^*) = 1$ for $T = 0$, whereas for $t=1$ the second formula gives $M_{3D}(1) = 0.9226036$, the value $t=1$ clearly being out of the indicated region. As $M_{3D}(1)$ should be 1, we remark that the formula given by Talapov and Blöte can be modified by adding a term

$$0.0773964.t^4$$

between the parentheses. This makes $M_{3D}(1) = 1$ and hardly influences the values of $M_{3D}(t)$ in the indicated region of validity. For $t=0.2$ the term gives about 0.0001. The

exponent 4 gave the most appropriate curve for the magnetization for small values of T, i.e. values of t close to 1. The comparison was carried out, however, without this modification.

The outcome was that the curves were clearly different, the simulating magnetization lying clearly above the one from Monte Carlo results. However, the exponent $\frac{1}{8}$ in the formula for $M_{2D}(H^*)$ was taken over from the 2D-setting, whereas the formula for the magnetization itself does not follow from $O_{2D}(H^*)$. In the light of the discussion of universality classes, to which we will come later, the critical exponent may be essentially different, for the 3D-model, from that for the 2D-model. We therefore replaced $\frac{1}{8}$ by $\frac{5}{16}$, often proposed in the literature, 0.32694109, the exponent given by Talapov and Blöte, and $\frac{3}{8}$, also often conjectured in the literature. We did not consider only the exponent 0.32694109, because in the interval $[0,0.0005]$, so very close to the critical point, the exponent still might increase to $\frac{3}{8} = 0.375$ when larger lattices are considered. We will come back to this later. It turned out that the best agreement occurred for the exponent $\frac{3}{8}$. The curves cross and in the region $[0.0005,0.26]$ for t differ at most 0.04. But this is too much in comparison with the accuracy of the Monte Carlo results. It is for this reason that we obtained the first indication that the simulation strength factor had to vary, be it slowly, for the temperature interval $[0,T_c]$.

2 Gauging on Transition Equations

The idea of gauging on equations is discussed in [6], in particular for the case of simulating the 2D+NNN-model by the 2D-model. NNN stands here for Next Nearest Neighbours. Including interactions between these leads to a non-planar lattice, for which, like for the non-planar 3D-lattice, no solution is known.

If $TE_{2D}(H^*) = 1$ is the transition equation for the 2D-model and $TE_{2D+NNN}(H, H_d) = 1$ is the transition equation for the 2D+NNN-model, where $H_d = \frac{J_d}{k_B T}$ and J_d is the isotropic interaction energy for "diagonal" interactions, then the gauging equation reads

$$TE_{2D}(H^*) = TE_{2D+NNN}(H, H_d).$$

This gives $H^*(H, H_d)$ and $O(H^*) = O(H, H_d)$ as solution for the 2D+NNN-model, once TE_{2D+NNN} is known. A good approximation to $TE_{2D+NNN}(H, H_d)$ was found by Zandvliet [16], leading to very good results for small values of H_d .

It is important to note that we must have that $TE_{2D+NNN}(H, 0)$ is identical with $TE_{2D}(H)$. The transition equations for a specific model can have various forms. This will be an important point in our later discussion.

For the 3D-model no exact transition equation is known either. From [6] we recall that

$$TE_{3D}(H_x, H_y, H_z) \equiv \sinh(2H_x)\sinh(2H_y) + \sinh(2H_x)\sinh(2H_z) + \sinh(2H_y)\sinh(2H_z) + 4(1 - \frac{1}{2}\exp[-4H_x - 4H_y - 4H_z])\sinh(2H_x)\sinh(2H_y)\sinh(2H_z) = 1$$

seems a good candidate. There is symmetry in the three variables. For H_x , H_y or H_z equal to zero, we recover the 2D-transition equation and, moreover, in the isotropic case where $H_x = H_y = H_z = H$, we obtain $H_c = 0.2216549$, in excellent agreement with the result from Monte Carlo calculations.

We should also note that in the isotropic case the equation reads

$$3\sinh(2H)^2 + 4\left(1 - \frac{1}{2}\exp[-12H]\right)\sinh(2H)^3 = 1,$$

whereas for the isotropic 2D-model

$$\sinh(2H^*)^2 = 1.$$

For $T \rightarrow 0$, H and H^* tend to infinity and $\sinh(2H) \approx \frac{1}{2}\exp[2H]$. Equating the left hand sides leads to

$$H^* \approx \frac{3}{2}H\left(1 + \frac{\ln 2}{6H}\right)$$

and we recover

$$\lim_{H \rightarrow \infty} f(H) = \frac{3}{2}.$$

Yet, we cannot simply, without further discussion, use $TE_{3D}(H) \equiv TE_{3D}(H, H, H)$ for gauging on the transition equations by

$$TE_{2D}(H^*) = TE_{3D}(H).$$

The reason for this is that the two models belong to different "universality classes".

3 Critical Exponents

It was discovered that there are relations between critical exponents of universal nature, see e.g. Widom [14], and internet for extensive literature. What is called the "scaling hypothesis" is the equation

$$\alpha + 2\beta + \gamma = 2.$$

Here α , β and γ are critical exponents for specific heat, magnetization and susceptibility, in the case of Ising models. Systems turned out to belong to "classes" for which very specific values of critical exponents are characteristic. For the 2D-Ising model the values are $\alpha_{2D} = 0$, logarithmic singularity, $\beta_{2D} = \frac{1}{8}$ and $\gamma_{2D} = \frac{7}{4}$. For the 3D-Ising model such exact values were not established yet.

We now consider our approach with focus on α_{3D} . The very fact that we simulate by the 2D-model implies that in first instance the logarithmic singularity for the specific heat

is taken over. The gauging by the equation $J^* = f(H).J$, with slowly varying $f(H)$, implies that the 3D-model has a logarithmic singularity. Our approach therefore implies

$$\alpha_{3D} = 0.$$

In the best of physical tradition our second argument comes from reference to experiments, on crystals that can be considered to be well modeled by the 3D-Ising model [10]. Experimental values satisfy

$$\alpha_{3D} \in [-0.02, 0.12], \beta_{3D} \in [0.38, 0.42], \gamma_{3D} \in [1.21, 1.33].$$

Monte Carlo calculations are very difficult to carry out close to the critical temperature. We already remarked that Talapov and Blöte do not make a statement about the interval $[0, 0.0005]$ for t . There is considerable consensus in the literature on $\gamma_{3D} = \frac{5}{4}$, well in the interval $[1.21, 1.33]$. Less consensus exists on β_{3D} . Kaupuzs [8] [9] or Zhang [17] conjecture $\beta_{3D} = \frac{3}{8}$, but others conjecture $\beta_{3D} = \frac{5}{16}$, see also Dalton and Wood [5].

With $\alpha_{3D} = 0$ and $\gamma_{3D} = \frac{5}{4}$ the scaling hypothesis gives

$$\beta_{3D} = \frac{3}{8},$$

which is, unlike $\beta_{3D} = \frac{5}{16}$, close to the experimental interval $[0.38, 0.42]$ (0.375 vs. 0.3125).

As scaling is assumed to hold across different physical systems it is interesting to note that in Berger *et al.* [2] for β_{3D} the range $[0.3649, 0.3715]$ is given for the 3D-Ising model and the range $[0.3620, 0.3670]$ for the 3D-Heisenberg model. Next to our findings in Section 1, we now have a second argument supporting $\beta_{3D} = \frac{3}{8}$.

4 On Transition Equations

Consider the formula

$$M_{2D} = [1 - E(y)]^{\frac{1}{8}},$$

where $y = \tanh(H^*)$ and $E(y_c) = 1$ for the critical value $y_c = \tanh(H_c^*)$. For $T \rightarrow 0$, $E(y) \rightarrow 0$. Here $E(y)$ is not yet specified further.

Usually an account of M_{2D} is given with $E(y) = \frac{T}{T_c}$. Now $E(y)=1$ is essentially a transition equation and with this form we can only conclude that $T = T_c$ is the critical temperature. Fortunately, we have the solution given by Yang [15], generalized by Chang [4] to

$$M_{2D} = [1 - (\frac{1}{\sinh(2H_x^*)\sinh(2H_y^*)})^2]^{\frac{1}{8}}.$$

So in Yang's formula

$$E(y) = \left(\frac{1}{\sinh(2H^*)}\right)^4 = \left(\frac{1}{TE_{2D}(y)}\right)^2$$

and $E(y)=1$ yields the true transition equation from which the critical point can be calculated.

Now we expect that the form of the formula in the isotropic 3D-case is

$$M_{3D} = [1 - U(x)]^{\frac{3}{8}},$$

where the exponent is considered to be settled, and $x = \tanh(H)$. $U(x)=1$ is again a transition equation.

As we have discussed in [6] the simulation yielded a solution function $y=S(x)$ that enabled to calculate a sequence of approximations to the critical point of the isotropic 3D-model. The equation

$$S(x) = \sqrt{(2)} - 1$$

is an approximation of a 3D-transition equation. By squaring we derive

$$(3 + 2\sqrt{(2)})S(x)^2 = 1.$$

We choose this form for $TE_{3D}(x)$ as we want to compare this equation with our candidate transition equation that starts with a term $3\sinh(2H)^2$, by our reasoning in [6] and that has a "correction" term that should vanish for H_x , H_y or H_z going to zero, reason why a factor $\sinh(2H_x)\sinh(2H_y)\sinh(2H_z)$ was, rather arbitrarily, chosen. The other factor in the second term was chosen to obtain the right transition point, but we stressed that no derivation of this candidate transition equation was given.

We consider the very first term, the one with x^2 in a development as power series in x . For the approximate transition equation in terms of $S(x)$ we find a first term

$$(3 + 2\sqrt{(2)}) \cdot \frac{9}{4} \cdot x^2 \approx 13.11x^2.$$

For the candidate transition equation we find a first term

$$3 \cdot 2^2 \cdot x^2 = 12x^2.$$

For this reason the candidate transition equation seems to fail. However, there is a serious additional problem. Finding a transition equation may give the correct transition point, but fail to give the right formula for $M_{3D}(H)$.

We will illustrate this point by considering the 2D-triangular Ising model with isotropic interactions. Various 2D-models and their solutions were given by Lin [11]. A simple transition equation is

$$TE_{TR,1} = 3\sinh(2H)^2 = 1,$$

giving $H_c = \frac{1}{4}\ln(3)$ for the transition point. However, for the magnetization the correct formula is

$$M_{TR} = [1 - (\frac{1}{TE_{TR,2}})^2]^{\frac{1}{8}},$$

where

$$TE_{TR,2} = 3\sinh(2H)^4 + 2\sinh(2H)^6 + 2\sinh(2H)^3\cosh(2H)^3.$$

$TE_{TR,2}$ is not a power of $TE_{TR,1}$, but $TE_{TR,2} = 1$ does give the correct value $H_c = \frac{1}{4}\ln(3)$.

So $TE_{TR,1}$ gives a correct transition equation, but does not lead to the right formula for M_{TR} . Something similar may hold for the 3D-model. One might derive a correct transition equation, yet not the correct formula for the magnetization. In $M_{3D}(H)$, $U(x) = \frac{1}{\sinh(f_c H)^4} = 1$ gives the correct transition equation $\sinh(f_c H) = 1$. Although the comparison with Monte Carlo results looks satisfactory for this choice of $U(x)$, it may yet not be the right $U(x)$ in the formula for M_{3D} .

The example of the triangular lattice shows that the right transition equation, i.e. one that not only gives the correct transition point but also occurs in the formula for M_{3D} , might look like (inspired by considerations in [6]) :

$$TE_{3D} = 3\sinh(2H)^4\cosh(2H)^2 + 8\sinh(2H)^6 + 6\sinh(2H)^3\cosh(2H)^3 = 1.$$

For H_c this equation gives $H_c = 0.2216379\dots$. In the case of an anisotropic lattice the first term reads

$$\begin{aligned} & \sinh(2H_x)^2\sinh(2H_y)^2\cosh(2H_z)^2 + \sinh(2H_x)^2\cosh(2H_y)^2\sinh(2H_z)^2 \\ & + \cosh(2H_x)^2\sinh(2H_y)^2\sinh(2H_z)^2. \end{aligned}$$

For $H_x = 0$, $H_y = 0$ or $H_z = 0$ the left hand side reduces to the Kramers-Wannier equation. The second and third term are symmetrical in the H 's as well and therefore vanish.

Gauging the 2D-transition equation on this 3D-transition equation gives

$$\sinh(2H^*)^4 = 3\sinh(2H)^4\cosh(2H)^2 + 8\sinh(2H)^6 + 6\sinh(92H)^3\cosh(2H)^3.$$

For $H \rightarrow \infty$ this equation also yields $\lim_{H \rightarrow \infty} f(H) = \frac{3}{2}$.

In view of the mentioned difficulty, that there is an (infinite) number of transition equations, that may even be of different structure, the best one can hope for is to find the right formula for the magnetization for the 3D-model. Not only would the right critical exponent β be found, but also the right transition equation. However, like for the 2D-model deriving the formula for the magnetization may be very hard.

5 On Gauging and Simulation

The idea of gauging on an equation, in particular on the transition equation, in order to arrive at a solution for the 3D-model, is seriously hampered by the facts discussed in Section 4. We had to make a distinction between a transition equation that gives the correct transition point, of which there are many (with $TE=1$ also $TE^n = 1$, n a positive integer greater than 1, is a transition equation), and one that also gives the right $U(x)$ in the formula for the magnetization, which we will call the *right* transition equation

We surmised in [6] that using the solution function $S(x)$ in a simulation gives a solution for the 3D-model that is rather close to the exact solution. This was put in serious doubt in Section 1. However we do have a result for the solution, using a gauging on the critical points, suggested by the equating of the two susceptibility series and the fact that the transition points seemed to correspond.

We also gave two arguments for the critical exponents. However, for the magnetization we assumed Yang's formula with the substitution of $f.H$ for H^* and then searched for the most likely exponent. But even if $\frac{3}{8}$ is the correct value of β_{3D} , then still $U(x)$ need not be $\frac{1}{\sinh(f(H)H)^4}$. Note that we now consider a function $f(H)$ instead of a constant f_c . We want to stress that searching for the right 3D-transition equation is still a must.

For dealing with the fact that the 2D-model and the 3D-model belong to different universality classes we focus again on the magnetization.

In Section 2 we saw that a good fit with Monte Carlo results for the magnetization was possible by raising M_{2D} to the power 3. The critical exponents depend on the dimensionality and in a simulation of the 3D-model by the 2D-model the difference in dimensionality means that one exponent does not follow from the other, i.e. the exponents are determined by the specific topological structure of the model and must be considered to be independent of each other. The way to take the gauging of universality classes into account, is to gauge by

$$M_{3D} = [1 - U(x)]^{\frac{3}{8}} = \left[\left[1 - \left(\frac{1}{TE_{2D}(y)} \right)^2 \right]^{\frac{1}{8}} \right]^3 = [1 - E(y)]^{\frac{3}{8}} = M_{2D}^{\frac{3}{8}}.$$

This then implies

$$U(x) = E(y).$$

as gauging equation.

The astonishing accuracy of the Monte Carlo results caused doubt about the simulating magnetization formula in which the *simulation strength factor* f was taken constant and equal to $\frac{H_{c,2D}}{H_{c,3D}}$. In Section 1 we derived that value from gauging on the critical temperature. For other values of the temperature T , or the variable H , of the 3D-model the simulation needs another simulation strength factor $f(H)$.

For the simulation strength factor we can make different assumptions when we want to see what our considerations about simulation have led to. In the f-H plane we assume a horizontal asymptote $f = \frac{3}{2}$ and the point (H_c, f_c) . We gave firm arguments for these two assumptions.

The slowly diminishing curve might be a hyperbole. However, as exponential functions dominate the whole Ising model, we assume an exponential decay :

$$f(H) = \frac{3}{2} + (f_c - \frac{3}{2})exp[-\phi(H - H_c)].$$

For the magnetization we choose

$$M_{3D} = [1 - \frac{1}{sinh(f(H).H)^4}]^{\frac{3}{8}}.$$

By equating this expression to the expression given by Talapov and Blöte, that we consider to give the "experimental" values, we find a simulation strength factor that is very close to 2 near the critical point and shows a decrease for T tending to zero. We therewith see the value for f_c confirmed. In Figure 1 the assumed simulation strength factor is plotted for $\phi = 1$ as a function of T.

Figure 2 shows the magnetization given by the formula of Talapov and Blöte, next to the magnetization given by the simulating formula M_{3D} with $\phi = 1$, the positive integer that gave the best fit.

Gauging on the right transition equation should give the theoretical way of determining f(H).

6 Discussion

The reader should be aware of the fact that we followed the approach outlined in [6]. We want to circumvent the bulk calculation and find potential solutions for free energy and magnetization, that can get a truth certificate by consistency with other considerations like series developments or experimental results.

The first result is the concept of simulation strength factor f(H) that enables to give the potential solution of the free energy of the isotropic 3D-problem as $O(f(H)H)$, where $O(H^*)$ is the 2D-solution given by Onsager. Two limit properties of f(H) were derived. The correct form of f(H) is, however, still an open problem.

We have given arguments for the values of the critical exponents and also have indicated how to take the difference in universality class into account when gauging. This led in particular to a potential solution for the 3D-magnetization. That formula is essentially Yang's formula with the exponent $\frac{1}{8}$ replaced by $\frac{3}{8}$.

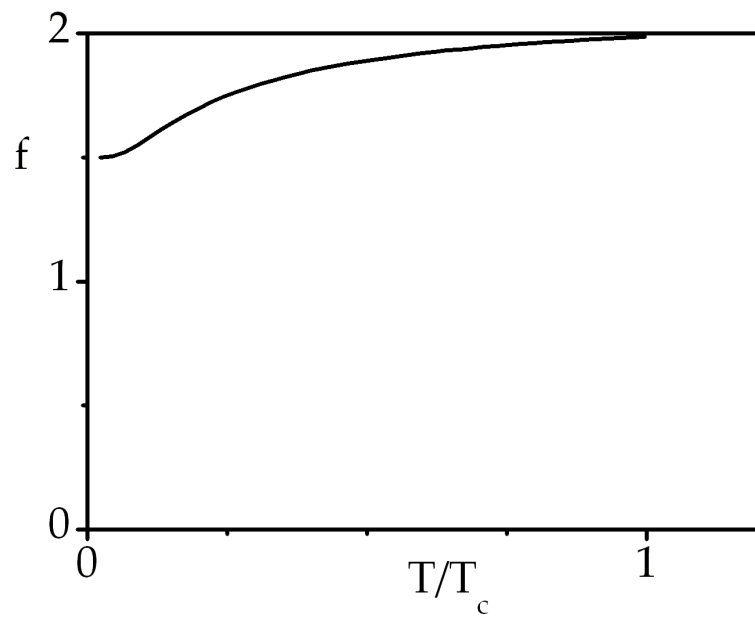


Figure 1: Simulation strength factor.

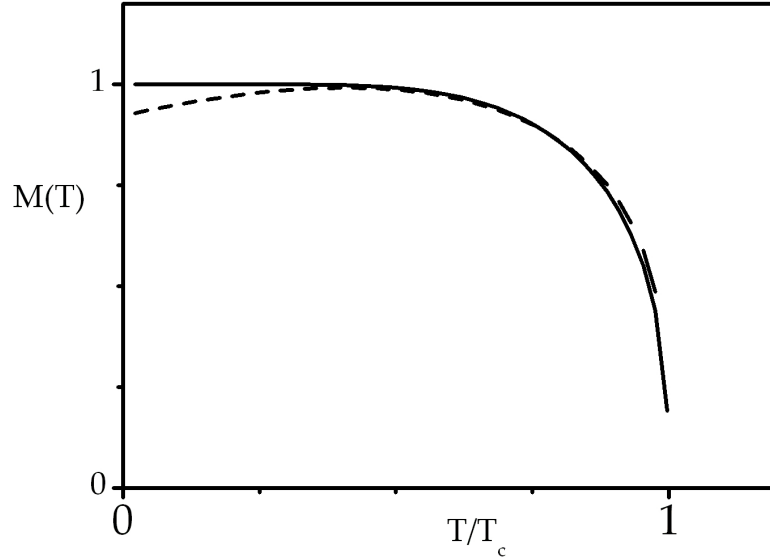


Figure 2: Two magnetization curves.

Our main conclusion is that simulation of the 3D-model by the 2D-model indeed seems possible, be it with a temperature dependent simulation strength factor $f(H)$, and that the quest for the definitive 3D-solution has been reduced to finding the right transition equation or, preferably, to finding the formula for the magnetization M_{3D} .

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