Approximate Solution of a Nonlinear Partial Differential Equation

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Abstract — Nonlinear partial differential equations (PDE) are notorious to solve. In only a limited number of cases can we find an analytic solution. In most cases, we can only apply some numerical scheme to simulate the process described by a nonlinear PDE. Therefore, approximate solutions are important for they may provide more insight about the process and its properties (stability, sensitivity etc.). The paper investigates the transient solution of a second order, nonlinear parabolic partial differential equation with given boundary- and initial conditions. The PDE may describe various physical processes, but we interpret it as a thermal process with exponential source term. We develop an analytical approximation, which describes the inverse solution. Accuracy and feasibility will be demonstrated. We also provide an expression for the time-derivative of the transient at time zero. The results can be extended for other boundary conditions as well.

Index Terms — distributed parameter systems, partial differential equations, heat processes, approximations.

I. INTRODUCTION

Partial differential equations play an important role in describing physical, industrial or biological processes. A large class of processes can be characterized by parabolic partial differential equations. These classes of processes include heat processes appearing in all kind of industrial or biological problems. A typical problem is to determine the transient temperature distribution of materials for given initial- and boundary conditions.

In many cases, there is a source term, which depends on some physical parameters. We consider a general case when the source term depends exponentially on the temperature. The process is thus described by a second order, non-linear partial differential equation. Typical examples are diffusion-reaction processes, some nuclear processes, chemical reactions or explosions, electric cables (thermal breakdown). It has also appeared in the theory of forming Nebulae (interstellar gas and dust) [17]. Although we interpret our process as a thermal process with internal source, the range of applications is wider. In retrospect, it is interesting to observe, that even a mechanical device called "Schmidt mechanisms" was developed in the 50's to plot approximate solutions of the heat equations [7].

Our aim is to analyze the transient behavior of the process. First, we establish the steady-state solution and its quadratic approximation, then develop a recursive formula for the inverse solution.

II. PROBLEM STATEMENT

Consider the following second order, parabolic partial differential equation (PDE) in dimensionless form:

\[
\frac{\partial T(\tau, z)}{\partial \tau} = \frac{\partial^2 T(\tau, z)}{\partial z^2} + g(T(\tau, z)) \quad \text{in} \ (-1,1) \times (0,\infty)
\]

(1)

where \( T(\tau, z) \) denotes temperature distribution, \( \tau \) denotes time in [r.u.], \( z \in \Omega \) where \( \Omega \) is a closed domain of the Euclidean space normalized to \( \Omega = [-1..1] \), and \( g(T(\tau, z)) \) denotes a source term which is a continuous function of the independent variable \( T(\tau, z) \). Equation (1) may describe various physical processes, as diffusion-reaction problems, electric space-charge problems, some nuclear process, explosions or forming a Nebulae or the temperature distribution in materials [3,8,9,17].

We define the process to be stable, if for a given source gain \( B \) the transient temperature \( T(\tau, z) \) reaches a steady-state value and the steady state is bounded. It is easy to understand that due to the exponential source term the process may become unstable. In earlier papers we established the stability conditions [13,14] and so we know that beyond a critical value of \( B > B_c \), no steady-state solution exists. However, if the process is stable \( (B < B_c) \), then it is of importance to say something about its transient behavior.

Next, we analyze the original PDE in time- and space and try to develop an approximate solution for a given boundary condition.
III. STEADY STATE SOLUTION

Let us now consider the steady state of (2) defined by:

\[
\frac{d^2T(z)}{dz^2} + B e^T(z) = 0
\]  

(3)

It is perhaps interesting to note that many different numerical techniques have been proposed to calculate the steady state solution of (2). It has somehow become a sort of benchmark to demonstrate accuracy and convergence of different numerical schemes. The most frequently used methods are finite difference, invariant imbedding, method of false transient and quasilinearization [2,8,9,10].

However, we have already established the general steady-state solution, which is given by [14]:

\[
T(z) = \ln \left( \frac{2\beta^2}{B \cosh^2(\beta(z-C_2))} \right)
\]  

(4)

where the unknown constants \( \beta \) and \( C_2 \) can be determined from the boundary conditions (the author has recently succeeded to determine the exact solution in two-dimensions as well; the result has not yet been published). Assume now that we have symmetric Dirichlet boundary conditions on \( z \in \partial \Omega \), where \( \partial \Omega \) denotes the boundary of \( \Omega \). Due to symmetry, it suffices to consider \( z \in \partial \Omega = \{0..1\} \) only, thus the BC’s are:

\[
\begin{align*}
z \in \partial \Omega & \quad T_z(0) = 0; \\
& \quad T(1) = T_1. 
\end{align*}
\]  

(5)

where \( T_z(z) = dT(z)/dz \) and \( T_1 = T(1) \). Without loosening generality we assume \( T_1 = 0 \) and zero initial condition \( T(0,z) = T_0(z) = 0 \). We conclude immediately that due to symmetry \( C_2 = 0 \) and \( T(z) \) can be rewritten into:

\[
T(z) = T_m - 2 \ln (\cosh(\beta z))
\]  

(6)

where \( T_m = T(0) \) denotes the maximum temperature appearing at \( z = 0 \) and parameter \( \beta \), which satisfies the boundary conditions, is:

\[
\beta = \text{arccosh} \left( e^{T_m/2} \right)
\]  

(7)

For any given gain \( B < B_1 \) (remember: for \( B > B_1 \) no steady-state solution exists) we can determine the corresponding maximum temperature \( T_m \). For symmetric Dirichlet BC’s we can derive the following relation between \( B \) and \( T_m \) in steady state:

\[
B = f(T_m) = 2 e^{-T_m} \text{arccosh}^2 (e^{T_m/2})
\]  

(8)

Figure 1 shows the function \( B = f(T_m) \). The critical (maximum) gain \( B_c \) can be determined from the condition \( \partial B/\partial T_m = 0 \). The numerical values are [14]:

\[
\begin{align*}
B_c &= 0,8784576797; \\
T_{mc} &= 1,1868421686; \\
\beta_c &= 1,1996786402.
\end{align*}
\]  

(9)

The author pointed out previously, that the steady-state equation has always two solutions as long as the source gain is less then the critical value \( B < B_c \) [14,16].

This can also be seen in Fig. 1 for the function \( B = f(T_m) \) is continuous and exists for \( T_m > T_{mc} \) as well.

Notice, that for every \( B < B_c \) there exist two different values of \( T_m \)! For example with \( B = 0.6 \) the two solutions are \( T_m = 0.55754 \) and \( T_m2 = 2,1626 \). It can be shown that only the solutions \( 0 < T_m < T_{mc} \) belong to a physically realizable system and therefore we call the solutions for \( T_m > T_{mc} \) virtual solutions. Due to this fact, care has to be taken if applying numerical methods: one must be sure that the numerical method converges to the true (realizable) solution and not to the virtual one.

IV. QUADRATIC APPROXIMATION OF STEADY-STATE

It is interesting to observe that the steady state solution given by (6) can be approximated by a quadratic (or parabolic) function, defined as:

\[
T_{p2}(z) = p_0 + p_1 z + p_2 z^2
\]  

(10)

where the unknown coefficients can be determined from the appropriate BC’s. This has long been known but earlier conclusions have been based on numerical simulations [6,13]. Using the exact solution given in (6) we investigate the accuracy of the approximation and show that it is indeed acceptably accurate.

For symmetric Dirichlet boundary conditions, the quadratic approximation can be expressed as\(^1\):

\[
T_{p2}(z) = T_m \left( 1 - v z^2 \right)
\]  

(11)

where \( v = 1 \).\(^2\) We define the pointwise error of the approximation as:

\[
e_{pw}(T_m,z) = T_{p2}(z) - T(z)
\]  

(12)

\(^1\) We must remark that a 3rd order approximation gives smaller approximation error. However, we loose an important property of the quadratic approximation, namely, \( T_p(z) \) is a monotone decreasing function which reflects the fact that heat is flowing from warmer to colder places. A 3rd order approximation - satisfying the BC’s - may not be monotonic.

\(^2\) With Robin boundary conditions the coefficients are: \( p_0 = T_m \), \( p_1 = 0 \), and \( p_2 = T_{mc} \alpha^*/(2+\alpha^*) \) where \( \alpha^* \) denotes the Biot-number.
Fig. 2 shows the pointwise error as a function of \( z \) and \( T_m \). It remains very small for smaller values of \( T_m \) and is increasing as \( T_m \) approaches its limit value \( T_m^c \). The maximum absolute error \( \max_{z} \| p(z) - T(z) \| \) appears around \( z \approx 0.7 \) almost independently of \( T_m \). We also define the L1- and L2-norm of the error:

\[
e_{L_1}(T_m) = \int_{\Omega} \left| p_2(z) - T(z) \right| d\Omega
\]

\[
e_{L_2}(T_m) = \left( \int_{\Omega} \left| p_2(z) - T(z) \right|^2 d\Omega \right)^{1/2}
\]

In Table 1 we give some values of the L1- and L2-norm of the error. In the Appendix, we give an analytical expression for the L1-norm, which shows why the error is small. We can thus conclude that the steady state solution can very well be approximated by a quadratic function.

<table>
<thead>
<tr>
<th>( T_m )</th>
<th>( e_{L_1} )</th>
<th>( e_{L_2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.000880</td>
<td>0.0011 \times 10^{-3}</td>
</tr>
<tr>
<td>0.4</td>
<td>0.003485</td>
<td>0.0173 \times 10^{-3}</td>
</tr>
<tr>
<td>0.6</td>
<td>0.007757</td>
<td>0.0853 \times 10^{-3}</td>
</tr>
<tr>
<td>0.8</td>
<td>0.013635</td>
<td>0.2628 \times 10^{-3}</td>
</tr>
<tr>
<td>1.0</td>
<td>0.021057</td>
<td>0.6252 \times 10^{-3}</td>
</tr>
<tr>
<td>1.1</td>
<td>0.025326</td>
<td>0.9032 \times 10^{-3}</td>
</tr>
</tbody>
</table>

Table 1. The L1- and L2-norm of the error defined by (13) and (14) for various values of \( T_m \).

V. APPROXIMATE TRANSIENT SOLUTION

Now we proceed to establish an approximate solution. Integrating each term of (2) and interchanging differentiation and integration leads to:

\[
\frac{d}{d\tau} \left\{ T_m \left( 1 - \frac{v}{3} \right) \right\} = B e^{T_m \sqrt{\frac{\pi}{2}}} \frac{\text{erf} \left( \sqrt{T_m} \right)}{\sqrt{T_m}} - 2vT_m
\]

(16)

where \( T_2(1) \) denotes the derivative of \( T(z) \) at \( z = 1 \), and \( \text{erf}(u) \) denotes the error function [1]. Separating the variables in (16) and by integrating again both sides we have \( v = 1 \):

\[
\tau = \int_0^\tau \frac{k_D}{B e^{T_m \sqrt{\frac{\pi}{2}}} \text{erf} \left( \sqrt{T_m} \right) - 2vT_m} dT_m
\]

(17)

where \( k_D = 1 - v/3 = 2/3 \) and the function \( f(.) \) is defined by:

\[
f(u) = \frac{\sqrt{\pi}}{2} \text{erf}(u)
\]

(18)

Recognize that (17) defines in fact the inverse solution of the transient behavior \( \tau = \tau(T_m) \). We can thus determine the time \( \tau \) necessary to increase the temperature from 0 to \( T_m \). Unfortunately, due to \( f(u) \) we can not evaluate the integral in closed form but we can calculate it for any value of \( T_m \). We’d like to note, that although the quadratic approximation \( T_p(z) \) is quite accurate, it gives a biased estimate concerning \( T(z) \) and its Taylor series around \( T_m = 0 \) (recall that \( \beta \) is defined by (7)) is:

\[
T_2(1) = -2T_m + \frac{1}{3}T_m^2 - \frac{1}{15}T_m^3 + O(T_m^4)
\]

(19)

Expressing the time increment necessary for the temperature to rise from \( T_n \) to \( T_{n+1} \) we can now rewrite (17) into a recursive form:

\[
\tau_{n+1} = \tau_n + \int_{T_n}^{T_{n+1}} \frac{k_D}{B e^{T_m \sqrt{\frac{\pi}{2}}} \text{erf} \left( \sqrt{T_m} \right) + T_2(1)} dt
\]

(20)

Once we know the inverse solution, the transient \( T(\tau,z) \) can be approximated by substituting the values of \( T_n \) into (11):

\[
T(\tau_n,z) = T_n(\tau_n) + \left( -vz^2 \right)
\]

(21)

VI. SIMULATION RESULTS

We have checked our analytic result with extensive numerical simulations. To simulate (2) we applied the Crank-Nicolson finite difference scheme [9,10] which is always numerically stable. Since the nonlinear function \( g(T) \) is differentiable, we can use Bellman’s quasilinearization technique [2], i.e. the function \( g(T) \) can be approximated at the grid point \( z = i\Delta z \) and time-level \( \tau_{n+1} = \tau_n + \Delta \tau \) as:
where prime denotes $\frac{\partial g(T)}{\partial T}$. Computer simulations revealed that a value of $\Delta z=0.01$ and $\Delta \tau=0.01$ usually guarantees sufficient numerical accuracy. We consider the result of the discrete scheme exact and compare it with our approximate solution. Consider now a process with gain $B = 0.6$. From (8) we can determine the corresponding steady state value of $T_m$, which is $T_m = 0.428$. Fig. 3 shows the root mean square error (RMSE) between the quadratic approximation and the exact solution per time step. Clearly, the RMSE is rather small indicating that $T(\tau,z)$ can indeed be approximated by a quadratic function during transient. Fig. 4 shows how well our analytic solution approximates the exact solution $z = 0$. Interestingly, we can establish the first derivative of the transient $T(\tau,z)$ even though no exact solution exists in analytic form:

$$g\left(T^{n+1}_i\right) = g(T^n_i) + \left(T^{n+1}_i - T^n_i\right) g'(T^n_i)$$

(22)

This is important for we can see that the gain $B$ governs the speed with which the transient begins to rise (or decrease) for all values of $z \in \Omega$ (the proof will be given in a more detailed paper).

Finally, Fig. 5 shows the complete transient for $z \in [0..1]$ and $\tau = [0..6]$ calculated by our approximate analytic formula. The calculation is very fast compared to the discrete scheme. Note, that to achieve high numerical accuracy with the discrete scheme, the number of grid points (in $z$) are about 100 which results in a 100x100 coefficient matrix which then must be inverted in every time step.

**CONCLUSIONS**

We have considered the transient behavior of a 2nd order, nonlinear partial differential equation with given boundary- and initial conditions. Having established the steady-state solution in closed analytic form, we give an accurate quadratic approximation. Based on the assumption that the profile of $T(\tau,z)$ does not change during transient, we have developed an approximate transient solution in a recursive form. The result is general and can be applied for different boundary conditions. We have also determined the speed with which the transient begins in closed form.

**REFERENCES**


**APPENDIX**

Consider (13) which defines the $L_1$-norm of the error $T_{p2}(z) - T(z)$. Notice, that both $T(z)$ and $T_{p2}(z)$ satisfies the boundary conditions, i.e. $T(0) = T_{p2}(0)$ and $T(1) = T_{p2}(1)$. We can drop the abs sign in the integral because $T_{p2}(z) \geq T(z)$ for $z \in \Omega$. We evaluate the integral of the $L_1$-norm to get:

$$e_{L_1}(T_m) = \int_0^{T_{p2}(z) - T(z)} dz =$$

$$= \left(2 \ln (\cosh(\beta z)) - \frac{\pi^2}{12}\beta + \frac{1}{\beta} \text{dilog}(1+e^{-2\beta}) \right) dz$$

$$= \frac{1}{3} T_m + \beta + 2 \ln(2) + \frac{\pi^2}{12\beta} + \frac{1}{\beta} \text{dilog}(1+e^{-2\beta})$$

(24)

where $\beta = \text{arccosh}(\frac{T_m}{2})$ and $\text{dilog}(.)$ denotes the dilogarithm function [1] (in spite of the negative sign the
function's value is positive. For the given domain of $T_m$ this is a continuous function.

We can calculate and plot the $L_1$-norm as a function of $T_m$ but instead, we determine its Taylor series around $T_m = 0$:

$$e_{L_1}(T_m) = \frac{1}{45}T_m^2 - \frac{1}{945}T_m^3 - \frac{1}{8100}T_m^4 + O\left(T_m^{9/2}\right)$$ (25)

This expression clearly reveals why the quadratic approximation of the exact steady-state solution is so accurate.

Figure 1. Steady state relation between gain $B$ and $T_m$ with symmetric Dirichlet BC's (dashed line denotes virtual solution).

Figure 2. Pointwise error $e_{pw}(T_m,z) = T_m(z) - T(z)$ with symmetric Dirichlet BC's.

Figure 3. Root Mean Square Errors (RMSE) between quadratic approximation and exact solution per time step.

Figure 4. Exact (continuous line) and approximate analytic solution (dashed line) at $x = 0$ with $B = 0.6$.

Figure 5. Transient solution calculated by the approximate analytic formula.