AN AVERAGE CASE ANALYSIS OF THE MINIMUM SPANNING TREE HEURISTIC FOR THE RANGE ASSIGNMENT PROBLEM

MAURITS DE GRAAF,* Thales Nederland B.V., and University of Twente

RICHARD J. BOUCHERIE, JOHANN L. HURINK, JAN-KEES VAN OMMEREN,**

University of Twente

Abstract

We present an average case analysis of the minimum spanning tree heuristic for the power assignment problem. The worst-case approximation ratio of this heuristic is 2. Our analysis yields the following results: (a) In the one-dimensional case, where the weights of the edges are 1 with probability \( r \), \( 0 < r < 1 \) and 0 otherwise, the expected approximation ratio for \( n \to \infty \) tends to \( 2 - r \). (b) In the one dimensional case, when the distances between neighboring vertices are drawn from a uniform \([0, 1]\)-distribution, the expected approximation ratio is bounded above by \( 2 - 2/(p + 2) \), where \( p \) denotes the distance power gradient. (c) When the edge weights are uniform \([0, 1]\) distributed, the expected approximation ratio is bounded above by 
\[
2 - 1/(2\zeta(3)),
\]
where \( \zeta \) denotes the Riemann zeta function. (d) In the Euclidean \( d \)-dimensional space, with distance power gradient \( 1 \leq p \leq d \) the power assignment \( W(P_n) \) converges completely (c.c.) to a constant \( \mu_P \leq 2\mu_Y \), where \( \mu_Y \) is the sum of the \( n/2 \) heaviest edges of the minimum spanning tree.

Keywords: average case analysis; range assignment; power assignment; ad-hoc networks; analysis of algorithms; approximation algorithms; point processes

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*Postal address: Thales Nederland B.V., Department Innovations, Research & Technology, P.O. Box 88, 1270 AB Huizen, Netherlands, e-mail: maurits.degraaf@nl.thalesgroup.com
**Postal address: University of Twente, Department of Applied Mathematics, P.O. Box 217, 7500 AE Enschede, Netherlands
1. Introduction

Ad hoc wireless networks have received significant attention in recent years due to their potential applications in battlefield, emergency disaster relief, and other scenarios (see, for example [17], [22], and [23]). In an ad hoc wireless network, a communications session is achieved either through single-hop transmission, if the recipient is within the transmission range of the source, or by relaying through intermediate nodes. The topology of a multihop wireless network is given by the set of communication links between node pairs. The topology depends on uncontrollable factors such as node mobility, weather, interference, noise as well as on controllable parameters such as transmit power. We assume an idealized propagation model, where omnidirectional antennas are used by all nodes to transmit and receive signals. We consider the case that for the purpose of energy conservation, each node can adjust its transmit power. For assigning the transmit powers, two conflicting effects have to be taken into account: if the transmit powers assigned to the nodes are too low, the resulting topology may be too sparse and the network may get partitioned. On the other extreme, if the transmit powers assigned to the nodes are too high, the nodes run out of energy quickly.

The goal of the Connected Minimum Power Assignment (CMPA-) problem is to assign transmit powers to the transceivers such that the resulting network is connected and the sum of transmit powers assigned to the transceivers is minimized (see e.g. [17]). This problem is, in general, NP-hard (for some special cases there are polynomial solutions). An intuitively appealing, and therefore well-known approximation approach is the Minimum Spanning Tree (MST-) heuristic. This heuristic is known to have a worst-case approximation ratio of 2 (see e.g. [15]). This paper presents an average case analysis of the MST-heuristic for the CMPA-problem.

1.1. Notation, previous work and contribution

For a set of points $V$ representing the nodes in a network, a power assignment can be represented as a function $p : V \rightarrow \mathbb{R}^+_0$. Following the notation of [17], for each ordered pair $(u, v)$ of transceivers, there is a transmit power threshold, denoted by $c(u, v)$, with the following meaning: a signal transmitted by the transceiver $u$ can be received by $v$ only when the transmit power $p(u)$ is at least $c(u, v)$. In this paper,
we assume that for each pair of points the values $c(u, v)$ are known and symmetric, i.e., $c(u, v) = c(v, u)$ for all pairs $\{u, v\} \in V$. Thus, a power assignment $p$ defines an undirected graph $G_p = (V, E_p)$, where $e = \{u, v\} \in E_p$ if and only if $p(u) \geq c(u, v)$ and $p(v) \geq c(u, v)$. Note that in the case $p(v) \geq c(u, v) > p(u)$ only transmission from $v$ to $u$ is possible. This might be interpreted as a unidirectional link from $v$ to $u$, but for most practical applications bidirectional (i.e., undirected) links are of interest: in wireless communications typically link-layer acknowledgements are involved, so only symmetric links are regarded.

This paper deals with the CMPA-problem: given a graph $G = (V, E, c)$, where $c$ denotes the edge weights $c : E \to \mathbb{R}^+$, one asks for a power assignment $p : V \to \mathbb{R}^+_0$ such that $G_p$ is connected and the total power $\sum_{v \in V} p(v)$ is minimal.

When $V \subset \mathbb{R}^d$, a power attenuation model is assumed, assuming that the signal power decreases with the distance $r$ as $r^{-\alpha}$, where the distance-power gradient $\alpha \in \mathbb{R}^+$ depends on the wireless environment and realistic values of $\alpha$ range from 1 to more than 6 [20]. This implies that $c(u, v) = r^\alpha$ if the distance between $u$ and $v$ is $r$. Therefore, in this case, the power assignment problem corresponds to assigning a range $r_v$ to node $v$, while asking for minimization of the sum $\sum_{v \in V} r_v^\alpha$. This is called the range assignment problem. Note that the power assignment problem is more general than the range assignment problem, as the weights need not be based on a distance function. While the main results of this paper relate to the range assignment problem, intermediate results are derived for the more general power assignment problem.

The range assignment problem is NP-hard in all dimensions $d \geq 2$ for all values of the distance-power gradient $\alpha$. The first NP-hardness result for the 3 dimensional range assignment problem was given in [15]. NP hardness in 2 dimensions was shown in [7]. In [4] and [11] the complexity of various other variants of the problem is analyzed.

Based on these complexity results, polynomial time approximation algorithms were studied. The first approximation algorithm to the CMPA-problem is the Minimum Spanning Tree (MST)-algorithm (see [6], [11]).
**MST Algorithm** $(V, E, c)$

1. Compute a minimum spanning tree $T$ using $c$ as edge costs.
2. For each node $v \in V$ assign $p(v) = \max\{c(e) | e \text{ incident to } v \in T\}$.

Let $T_n$ denote a minimum spanning tree of a graph on $n$ vertices. In addition, let $P_{T_n}$ denote the power assignment corresponding to $T_n$, i.e., for each $v \in V$: $P_{T}(v) = \max\{c(e) | e \in T \text{ and } e \text{ incident to } v\}$. When it is clear from the context which $T_n$ is meant, we simply write $P_n$ instead of $P_{T_n}$. We define $W(T_n)$ to be the total weight of the spanning tree $T_n$, and $W(P_n)$ the total weight of the corresponding power assignment. It is well established (see e.g. [3], [6]) that

$$W(T_n) \leq W(P) \leq W(P_n) \leq 2W(T_n) \quad (1)$$

where $W(P)$ denotes the weight of the optimal power assignment $P$. In [3] it is shown that the factor 2 is tight.

For the MST algorithm, (1) shows that the worst-case performance ratio is 2. Other approximation algorithms are studied in [3], where a polynomial time approximation scheme with a worst-case performance ratio approaching $5/3$ as well as a more practical approximation algorithm with a worst-case approximation factor of $11/6$ are given.

While the worst-case performance ratio of 2 might discourage use of the MST algorithm in practice, numerical results indicate that the MST algorithm is often rather close to the optimal solution [24]. Besides these numerical observations, the average case behavior of the MST algorithm has never been analyzed. However, a probabilistic analysis of the range assignment problems has been performed in [24] focusing on upper- and lower-bounds for connectedness in the special case that all nodes have the same transmit power.

**Statement of contribution.** This paper presents an analysis of the average case behavior of the function $W(P_n)/W(T_n)$ for $n \to \infty$, which provides an upper bound to the performance ratio $W(P_n)/W(P)$. We investigate the following situations: for $V \subset \mathbb{R}^d$, we provide an upper bound for $d = 1$, and we derive a general theoretical result relating the upper bound to the minimum spanning tree constant for $1 < p \leq d$. We also investigate the situation for the complete graph with independent uniform
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edge weights, where the expected approximation ratio turns out to have a closed form upper bound.

The paper is organized as follows. In Section 2 we provide some preliminary results. Section 3 analyzes the 1-dimensional case for edge weights in the set \{0, 1\} and for uniformly distributed edge weights on [0, 1]. Section 4 presents results for the d-dimensional case where \( d \geq 2 \). Finally, Section 5 presents conclusions and directions for further research.

2. Preliminaries

2.1. Minimum spanning trees

Let \( G = (V, E) \) be a graph with \(|V| = n\) and a cost function \( c : E \rightarrow \mathbb{R}^+ \). Furthermore, for a vertex \( v \), let \( G\{v\} \) denote the graph arising from \( G \) by deleting \( v \) and all edges incident to \( v \), for an edge \( e \), \( G\{e\} \) denotes the graph arising from \( G \) by deleting edge \( e \). Suppose let \( F = (V, E_F) \) is a forest on \( G \)(i.e., a graph with no cycles) with \( E_F = \{e_1, \ldots, e_m\} \subseteq E \). We assume \( c(e_1) \leq c(e_2) \leq \ldots \leq c(e_m) \) and let \( S_k(F) = \sum_{j=m-k+1}^{m} c(e_j) \) denote the sum of the \( k \) heaviest edges of \( F \), for \( k \in \{1, \ldots, m\} \). If \( m = n - 1 \), then \( F \) is a spanning tree, which we denote by \( T \). For a given tree \( T \), we say that an edge \( e \) incident to \( v \) covers \( v \), if (a) \( e \in T \), i.e., \( e = e_i \) for some \( i \in \{1, \ldots, n - 1\} \), and (b) the index \( i \) is maximal among the edges \( e \) incident to \( v \). Note that condition (b) ensures that each vertex is covered by exactly one edge and that \( c(e_i) \geq c(e_j) \) for all edges \( e_j \in T \) incident to \( v \).

Let \( f(e) \) denote the number of nodes covered by \( e \in T \), called the \textit{covering number} of \( e \in E \). Note that \( f(e) \in \{0, 1, 2\} \). We immediately see that \( \sum_{e \in E} f(e) = n \) as each vertex is covered exactly once. Moreover, \( W(P_n) = \sum_{e \in T} f(e)c(e) \).

The following observation strengthens (1).

\textbf{Lemma 2.1.} Let the edges \( e_1, \ldots, e_{n-1} \) of a minimum spanning tree \( T \) be sorted such that \( c(e_1) \leq c(e_2) \leq \ldots \leq c(e_{n-1}) \). Then

\[ c(e_{n-1}) + S_{n-1}(T) \leq W(P_T) \leq \begin{cases} c(e_{\lfloor n/2 \rfloor}) + 2 S_{\lfloor n/2 \rfloor}(T) & \text{if } n \text{ is odd,} \\ 2 S_{n/2}(T) & \text{if } n \text{ is even,} \end{cases} \quad (2) \]
Proof. The right-hand inequalities of (2) follow from the fact that \( f(e) \in \{0, 1, 2\} \) and \( \sum_{e \in E} f(e) = n \), and therefore \( W(P_T) = \sum_{i=1}^{n-1} f(e_i) c(e_i) \) takes its maximum, when \( f \) takes maximum values for the edges with the highest weights. The left-hand inequality can be inferred by induction on \( n \): for \( n = 2 \) the statement is clearly true. For \( n \geq 2 \), suppose the inequality is true for all trees \( T' \) on \( n - 1 \) vertices. Now, let \( T \) be a tree on \( n \) vertices. As there are at least two edges in \( T \) that are incident to a vertex of degree 1 (in \( T \)), there exists an edge \( e_k \), \( k < n - 1 \), which is incident to a vertex \( w \), which has degree 1 (in \( T \)). As \( e_k \) covers \( w \), we have \( f(e_k) \geq 1 \). Let \( G' := G \setminus \{w\} \), and consider \( T' = T \setminus \{w\} \). By the choice of \( e_k \), \( T' \) is a minimum spanning tree of \( G' \) and by the induction hypothesis \( W(P_{T'}) \geq c(e_{n-1}) + S_{n-2}(T') = c(e_{n-1}) + \sum_{i=1, i \neq k}^{n-2} c(e_i) \). So that \( W(P_T) \geq W(P_{T'}) + c(e_k) \). This completes the proof. □

The following example shows that the bounds for the inequalities (2) are tight.

Example 2.1. Let \( G = (V, E) \) be a path \( e_1, \ldots, e_n \) such that \( c(e_j) = 1 \) if \( j \) is odd, and \( c(e_j) = \epsilon < 1 \) if \( j \) is even. \( G \) has only one spanning tree \( T \) which is equal to the graph itself: \( T = G \). Sorting the edges according to increasing costs we first obtain \( \lceil (n - 1)/2 \rceil \) edges of cost \( \epsilon \), followed by \( \lfloor (n - 1)/2 \rfloor \) edges of cost 1. Moreover, \( W(T) = \lceil (n - 1)/2 \rceil + \lfloor (n - 1)/2 \rfloor \epsilon \). Clearly, all edges with an odd index have covering number 2, and, if \( n \) is odd, say \( n = 2m + 1 \), there is only one edge (being \( e_{2m} \)) with covering number 1, incident to the last vertex. So \( W(P_T) = 2m + \epsilon \), which is a tight bound for (2) for odd \( n \). If \( n \) is even, say \( n = 2m \) then \( W(P_T) = 2m \), which is a tight bound for (2) for even \( n \). An example for tightness of (2) is obtained by considering a graph \( G = (V, E) \) where all costs \( c(e) \) are 1. In this case \( W(T) = n - 1 \) and \( W(P_T) = n \). □

Lemma 2.1 implies,

\[
W(P_T) \leq 2S_{\lceil n/2 \rceil}(T).
\] (3)

Next, we state a relation between the minimum spanning trees of a graph and its extension by a single vertex, and additional edges between the new vertex and the existing vertices. The following well-known extension property (see e.g., [14]) for spanning trees plays a central role. Let \( T_1 \) and \( T_2 \) be two spanning trees on \( V \). Then for each \( e \in T_1 \) there is an \( f \in T_2 \) so that \( T_1 \setminus \{e\} \cup \{f\} \) is again a spanning tree.
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Note that Lemma 2.2 relates to the construction in [16] and [14], where it is shown how a forest in $G(V\{v'\})$ can be extended to a minimum spanning tree using the ‘add and drop’ algorithm. Their proof, however, does not consider the upper bound on the weights $w(e)$, nor seems easily extensible in this way, while the lemma below can be extended to Lemma 5.1.

**Lemma 2.2.** Let $G = (V, E)$ be a graph with $|V| = n$ and let $H = (V \cup \{z\}, E \cup E')$ be an extension of $G$, where $E' = \{(v, z) \mid v \in V\}$. Furthermore, let $T(H)$ be a minimum spanning tree on $H$ and let $F = T(H) \setminus \{z\}$. Then there exists a minimum spanning tree $T'$ of $G$, with $F \subseteq T'$.

**Proof.** Let $c : E \cup E' \rightarrow \mathbb{R}^+$ be the weight function on the edges and let $T'$ be a minimum spanning tree on $G$ with the property that the number of edges in $T' \cap F$ is maximal. Assume there exists an edge $e = \{u, w\} \in F \cap T' = T(H) \setminus (\{z\} \cup T')$. As $e \in T(H)$ it follows that $T(H) \setminus \{e\}$ consists of two components, say, $K_1$ and $K_2$. Let $P$ denote the path in $T'$ connecting $u$ and $w$. $P$ must contain an edge $f = \{x, y\}$ with $x \in K_1$ and $y \in K_2$. Now $T(H) \setminus \{e\} \cup \{f\}$ is a spanning tree of $H$, and consequently $c(e) \leq c(f)$. As $f \in P$, also $T' \setminus \{f\} \cup \{e\}$ is a spanning tree of $G$, and we get $c(e) \geq c(f)$. It follows that $c(e) = c(f)$, and thus $T' \setminus \{f\} \cup \{e\}$ is also a minimum spanning tree of $G$, but with a larger intersection with $F$. This contradicts the choice of $T'$ and therefore no edge $e \in F \setminus T'$ exists, implying that $F \subseteq T'$ and completing the proof. □

3. One dimension: the Spanning Tree is a path

3.1. Weights in the set $\{0, 1\}$

Let $G = (V, E)$ be a path of $n$ vertices, with $n > 0$, and where the cost $c(e)$ of an edge $e \in E$ is 1 with probability $s$ and 0 with probability $1 - s$. Note that in $T_n$ is equal to $G$. In this case, $W(T_n)$ and $W(P_n)$ are r.v.'s denoting the total weight of the minimum spanning tree, and the total weight of the power assignment corresponding to $T_n$, respectively.

The weight $W(P_n)$ of the MST approximation of the power assignment problem depends on the distribution of the 1’s over the path. We define a run as a succession of 1’s preceded and succeeded by 0’s. The number of elements in a run is referred to
as its length. Let $R_i$ denote the r.v. indicating the number of runs of length $i$. In addition, let $R$ and $N$ be the r.v.’s indicating the total number of runs and the number of edges with weight 1. (Note that $R \leq N$.)

**Example 3.1.** (Illustration of definition of runs.) Let $n = 11$, so $G$ is a path of 10 edges. Both 0101010101 and 1111100000 are possible weight assignments with $W(T_n) = 5$. For the first series, the associated power assignment has weight $W(P_n) = 10$ for the second series the associated power assignment has weight $W(P_n) = 6$. Both series have $N = 5$. The number of runs $R_1 = 5$ for the first path and for the second $R_5 = 1$.

We prove the following theorem:

**Theorem 3.1.** Let $G = (V, E)$ be a graph as described above. Then

$$E \left[ \frac{W(P_n)}{W(T_n)} \right]_{N > 0} = 1 + \frac{(1 + \frac{1}{n})(1 - (1 - s)^n) - s}{1 - (1 - s)^n}.$$ 

Proof. As $T_n$ equals $G$, we have $W(T_n) = N$. As each run of 1’s of length $i$ contributes $(i + 1)$ to the power assignment, we have:

$$W(P_n) = \sum_{i=1}^{n} R_i(i + 1) = \sum_{i=1}^{n} i R_i + \sum_{i=1}^{n} R_i = N + R.$$ 

The conditional distribution of $R$, given that $N = n_1$, was derived by Mood in [18]:

$$P(R = r|N = n_1) = \binom{n_1 - 1}{r - 1} \binom{n - n_1 + 1}{r} \binom{n}{n_1}.$$ (4)

Based on this, for the expected number of runs, given $N = n_1$, we get: $E[R|N = n_1] = \binom{n - n_1 + 1}{n_1} n$. To calculate $E[R/N]$, assume $n_1 > 0$. We obtain $E[R/N|N = n_1] = \binom{n - n_1 + 1}{n}$, and, using the fact that $N \sim \text{Binomial}(n, s)$,

$$E[R/N \cdot 1_{N_1 > 0}] = \sum_{n_1=1}^{n} E[R/N|N = n_1]P(N = n_1)$$

$$= \frac{1}{n} + 1 - s - (1 + \frac{1}{n})(1 - s)^n.$$ 

Observe that $W(P_n)/W(T_n) = 1 + R/N$. This completes the proof. □

This result can be intuitively explained as follows. When $p > 0$ is very small, the runs are most likely of length 1, in this case $W(P_n) = 2W(T_n)$. On the other
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extreme, when \( s \) is close to 1, most likely there is a single run of 1’s, in which case
\[ W(P_n) = W(T_n) + 1. \]
Assume \( n \) is large and \( 0 < s < 1 \). Then \( (1 + 1/n)(1 - (1 - s)^n) \)
is close to 1, so \( E[W(P_n)/W(T_n)|N_1 > 0] \approx 2 - s \). While the corresponding limit
result also follows from Theorem 3.2 in the next section, we believe the exact result is
interesting as it allows to get insight in the approximation quality of the limit result.

3.2. Uniformly distributed weights

Next, we consider the situation where \( G = (V, E) \) is a path of \( n \) vertices \( X_1 \leq \ldots \leq X_{n+1} \in \mathbb{R}^1 \) where the transmit power thresholds \( D_i = X_{i+1} - X_i \) to connect
neighboring vertices are i.i.d. nonnegative random variables with finite expectation.

**Theorem 3.2.** Let \( G = (V, E) \) be a path as defined above. Then
\[
\frac{W(P_n)}{W(T_n)} \xrightarrow{a.s.} \frac{E[\max\{D_1, D_2\}]}{E[D_1]}. \tag{5}
\]

*Proof.* We get

\[
W(P_n) = D_1 + \sum_{i=1}^{n-1} \max\{D_i, D_{i+1}\} + D_n
= D_1 + \sum_{i=1}^{\lfloor n/2 \rfloor} \max\{D_{2i-1}, D_{2i}\} + \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} \max\{D_{2i}, D_{2i+1}\} + D_n.
\]

Since, by splitting the odd and even terms, in both sums the random variables are
i.i.d., it follows by the strong law of large numbers, that
\[
\frac{W(P_n)}{n} \xrightarrow{a.s.} E[\max\{D_1, D_2\}].
\]

Being the sum of i.i.d. r.v.’s, \( W(T_n) \) also satisfies the strong law of large numbers, and
we obtain:
\[
\frac{W(T_n)}{n} \xrightarrow{a.s.} E[D_1],
\]
implying almost sure convergence of \( \frac{W(P_n)}{W(T_n)} \). \( \square \)

Since, \( W(P_n) \) and \( W(T_n) \) are sequences of r.v.’s, with bounded ratio, the above
theorem implies that these ratios also converge in mean and that \( E[W(P_n)/W(T_n)] \)
converges. So we have:
Corollary 3.1.

\[
\lim_{n \to \infty} \frac{\mathbb{E}[W(P_n)]}{\mathbb{E}[W(T_n)]} = \frac{\mathbb{E} [\max\{D_1, D_2\}]}{\mathbb{E}[D_1]}
\]

Next, we consider the specific situation of \(n\) vertices in \(\mathbb{R}^1\) with transmit power threshold \(D_i \sim U^p\), with \(U \sim U[0,1]\), where \(U[0,1]\) denotes the uniform distribution, and \(p\) models the distance-power gradient.

Corollary 3.2. If in the situation of Theorem 3.2, we have for \(i = 1, \ldots, n\), \(D_i \sim U^p\), with \(U \sim U[0,1]\) then

\[
\frac{W(P_n)}{W(T_n)} \xrightarrow{a.s.} 2 - \frac{2}{p+2}.
\]

Proof. It easily follows that \(\mathbb{E}[D_1] = \frac{1}{p+1}\), and \(\mathbb{E} [\max\{D_1, D_2\}] = \frac{2}{p+2}\). \(\square\)

Remark. For the model of the previous section, we obtain by the same techniques that

\[
\frac{W(P_n)}{W(T_n)} \xrightarrow{a.s.} 2 - p.
\]

Since the two terms \(W(P_n)\) and \(W(T_n)\) are bounded, it follows that \(\mathbb{E}[W(P_n)/W(T_n)] \to 2 - p\).

4. The complete graph with uniformly distributed weights

In this section we consider a complete graph with uniformly distributed edge weights and apply Lemma 2.1 to this graph. More precisely, let \(G = (V, E, \tilde{X})\) be a graph with uniformly distributed edge weights, where \(V = \{1, 2, \ldots, n\}\) and \(\tilde{X} = \{X_{ij} : 1 \leq i < j \leq n\}\) a sample of size \(\binom{n}{2} = \binom{n}{2}\) with \(X_{ij} \sim U[0,1]\), describing the edge weights.

Let \(Y_n = Y_n(G)\) denote the set of \(\lfloor n/2 \rfloor\) most expensive, and \(U_n = U_n(G)\) the set of \(\lfloor n/2 \rfloor\) least expensive edges of \(T_n\), and \(W(Y_n), W(U_n)\) the weight of \(Y_n, U_n\). Note that \(W(Y_n) = S_{\lfloor n/2 \rfloor}\).

In order to analyze \(W(Y_n)\) and \(W(U_n)\), we follow the exposition of Frieze’s result (Theorem 4.1) of [5]. Let \(X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(\binom{n}{2})}\) denote the order statistics of the sample \(\tilde{X} = (X_{ij})\). With probability 1 we have \(X_{(1)} < X_{(2)} < \ldots < X_{(\binom{n}{2})}\). Now \(\tilde{X} = (X_{ij})\) defines a graph process \(G_t\) in a natural way, in which edges are added over time, where the edge set of \(G_t\), \(0 \leq t \leq \binom{n}{2}\), is given by the \(t\) least expensive edges of \(G\):

\[\{\{i,j\} : X_{ij} = X_{(k)} \text{ for some } k \leq t\}.\]
Given $G_t$, define the r.v. $\psi(\cdot)$ as:

$$\psi(k) = \min\{t : w(G_t) = n - k\}, k = 1, 2, \ldots, n - 1,$$  \hspace{1cm} (7)

where $w(G_t)$ denotes the number of components of $G_t$. Then $\psi(1) = 1$, $\psi(2) = 2$ and $\psi(n - 1)$ is the first time $t$ when the graph $G_t$ is connected. Clearly $\psi(k) \geq k$. By the greedy algorithm of Kruskal, a minimum spanning tree of $G$ is formed by the edges of $G_t$ appearing at times $\psi(1), \psi(2), \ldots, \psi(n - 1)$, i.e.,

$$W(T_n) = \sum_{k=1}^{n-1} X_{\psi(k)}.$$

By a theorem of Frieze [10] we have:

**Theorem 4.1.** ([10],) Let $G = (V, E, \tilde{X})$ be a complete graph, with uniformly distributed edge weights. Then the weight of the minimum spanning tree $T_n$ satisfies,

$$W(T_n) \xrightarrow{P} \zeta(3),$$  \hspace{1cm} (8)

where $\zeta(\cdot)$ denotes the Riemann zeta function.

We show:

**Lemma 4.1.** Let $G = (V, E, \tilde{X})$ be a complete graph, with uniformly distributed edge weights. Then for $W(U_n) = \sum_{k=1}^{\lfloor n/2 \rfloor} X_{\psi(k)}$:

$$W(U_n) \xrightarrow{P} 1/4.$$  \hspace{1cm} (9)

**Proof.** The idea behind the proof of statement (9) is that, for large $n$ and $1 \leq k \leq n/2$, $\psi(k)$ is ‘close’ to $k$. In other words: the least expensive edges of large graphs with uniformly distributed edge weights do not contain a circuit. First we show

$$\lim_{n \to \infty} \mathbb{E}[W(U_n)] = 1/4.$$  \hspace{1cm} (10)

Clearly, as $\mathbb{E}[X(i)] = i/(\binom{n}{2} + 1)$, we have

$$\mathbb{E}[W(U_n)] = \sum_{k=1}^{\lfloor n/2 \rfloor} \mathbb{E}[\psi(k)]/(\binom{n}{2} + 1).$$

Since $\psi(k) \geq k$, one easily finds $\mathbb{E}[W(U_n)] \geq 1/4$. We will show equality.

First we note:

$$\mathbb{P}(k \leq \psi(k) \leq k + n^{8/9}) = 1 - o(n^{-1/4}).$$  \hspace{1cm} (10)
This follows from (6.18) in [5] which states that, with probability $1 - o(n^{-1/4})$,

$$k \leq \psi(k) \leq \psi_0(k) + n^{8/9}, \text{ for } k = 1, \ldots, \lfloor n/2 \rfloor,$$  
(11)

where $\psi_0(k)$ is defined by:

$$u\left(\frac{2\psi_0(k)}{n}\right) = 1 - \frac{k}{n},$$  
(12)

where

$$u(x) = \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} x^{k-1} e^{-kx}.$$

Now, from the fact that $u$ is a one-to-one mapping from $\mathbb{R}^+$ to $(0, 1)$, and (see [5], page 109) that for $0 \leq x \leq 1$: $u(x) = 1 - x/2$, it follows that: $\psi_0(k) = k$, for $0 \leq k \leq n/2$.

Now (10) implies:

$$k \leq E[\psi(k)] \leq k + n^{8/9} + o(n^{3/4} \log n)$$  
(13)

This follows from (6.14) in [5] which states, $P(\psi(n-1) \leq 2n \log n) = 1 - O(n^{-3})$. So that

$$k \leq E[\psi(k)] \leq (k + n^{8/9})P(k \leq \psi(k) \leq k + n^{8/9}) + 2n \log n P(k + n^{8/9} \leq \psi(k) \leq 2n \log n) + \begin{pmatrix} n \\ 2 \end{pmatrix} P(2n \log n \leq \psi(k) \leq \begin{pmatrix} n \\ 2 \end{pmatrix}),$$

which implies $k \leq E[\psi(k)] \leq k + n^{8/9} + o(n^{3/4} \log n) + O(n^{-1})$.

By (13) we obtain,

$$\lim_{n \to \infty} E[W(U_n)] = \lim_{n \to \infty} \frac{1}{\begin{pmatrix} n \\ 2 \end{pmatrix}} + 1 \sum_{k=1}^{\lfloor n/2 \rfloor} E[\psi(k)] = \frac{1}{4}$$  
(14)

Which shows (9).

To see convergence in probability, we remark that in [10] equation (15) it is shown that:

$$E[X_{\psi(k)}X_{\psi(l)}] = \frac{E[\psi(k)\psi(l)] + E[\psi(l)]}{\left(\begin{pmatrix} n \\ 2 \end{pmatrix} + 1\right)\left(\begin{pmatrix} n \\ 2 \end{pmatrix} + 2\right)},$$

so in order to show that $\text{Var}[W(U_n)] \to 0$ it suffices to show that

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{l=1}^{\lfloor n/2 \rfloor} E[\psi(k)\psi(l)] \leq (1 + o(1)) \sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{l=1}^{\lfloor n/2 \rfloor} E[\psi(k)E[\psi(l)]]$$

This follows from [10] equation (17) which implies that for some constant $c$, and $1 \leq k \leq l \leq n/2:

$$E[\psi(k)\psi(l)] \leq E[\psi(k)]E[\psi(l)] + cn^{11/6}(\log n)^2$$
Now, we obtain (9) from Chebyshev’s inequality. □

This leads to the following theorem for power assignments.

**Theorem 4.2.** Let \( G = (V, E, \bar{X}) \) be a complete graph, with uniformly distributed edge weights. Then,

\[
\limsup_{n \to \infty} \frac{E[W(P_n)]}{W(T_n)} \leq \frac{2(\zeta(3) - 1/4)}{\zeta(3)} \approx 1.58...
\]

**Proof.** As \( W(T_n) \leq W(P_n) \leq 2W(Y_n) \), and \( W(T_n) = W(U_n) + W(Y_n) \), it follows from Theorem 4.1 and Lemma 4.1 that \( \zeta(3) \leq \limsup_{n \to \infty} E[W(P_n)] \leq 2(\zeta(3) - 1/4) \).

\( \square \)

5. Convergence results for the power assignment, general \( 1 \leq p \leq d \).

In this section we consider complete graphs \( G(V) \) on sets \( V \subseteq \mathbb{R}^d \), with \(|V| = n\), where the vertices represent the nodes in the communication network. If the number of points needs to be made explicit, we write \( V_n \) instead of \( V \). The weight of an edge is given by \( c(\{x, y\}) = \text{dist}(x, y)^p \), where ‘dist’ denotes the Euclidean distance. We denote by \( T(V) \) the weight of a minimum spanning tree on \( G(V) \), and assume that there is always a unique minimum spanning tree. Furthermore, \( P(V) \) denotes the weight of the power assignment \( P_{T(V)} \) resulting from application of the MST algorithm on \( V \). For brevity, we write \( S_k(V) \) for the weight of the \( k \) heaviest edges of the minimum spanning tree on \( V \). So, referring back to the notation in Section 2, \( P(V) = W(P_{T(V)}) \), and \( S_k(V) = S_k(T(V)) \). The \( P(V) \) and \( S_k(V) \) functionals are Euclidean functionals according to Yukich [30]. Based on results from Yukich [30] we derive bounds on the ratio \( P(V)/T(V) \).

Given a hyperrectangle \( R \subseteq \mathbb{R}^d \), let \( P_B(V, R) \) denote the weight of the power assignment according to the **boundary MST functional** (see [31], page 14). That is, \( P_B(V, R) \) is obtained by first calculating the boundary spanning tree on \( R \), and then applying the MST power assignment algorithm to assign powers to all \( v \in V \). Note that in the boundary functional, no power is assigned to the boundary. The boundary power assignment functional inherits superadditivity (see [30] (3.3)) from the boundary minimum spanning tree. So \( P_B(F, R) \geq P_B(F, R_1) + P_B(F, R_2) \) for every partitioning of \( R \) into hyperrectangles \( R_1 \) and \( R_2 \). Note that when \( P(V, R) \neq P_B(V, R) \), the spanning
forest realizing the boundary functional $P_B(V,R)$ may be thought of as a collection of small trees $T_1, \ldots, T_Q$ connected via the boundary of $R$ into a single large tree, where the connections on the boundary of $R$ incur no cost. In such a case, each $T_i$, $i = 1, \ldots, Q$, is a minimum spanning tree.

Following [30] (3.8) we call $P(\cdot)$ smooth if for all $V$, $W$: $P|(V \cup W) - P(V)| = O(|W|^{(d-p)/d})$. We say $P(V)$ and $P_B(V)$ are pointwise close if $|P(V) - P_B(V)| = o(1)$. As in [30] (4.11), we call $P(V)$ close in mean to $P_B(V)$ if $E[|P(U_1, \ldots, U_n) - P_B(U_1, \ldots, U_n)|] = o(n^{(d-p)/d})$. We call $P(\cdot)$ smooth in mean (as defined in [30] (4.13)) if there exists a constant $\gamma < 1/2$ such that for all $n \geq 1$ and $0 \leq k \leq n/2$ we have $E[|P(U_1, \ldots, U_n) - P(U_1, \ldots, U_n_{\pm k})|] \leq Ckn^{(d-p)/d+\gamma}$. To formulate our result, we also need the notion of complete convergence. A sequence of r.v.'s $\{X_n\}_n$, converges completely (c.c.), (notation: $X_n \overset{c.c.}{\to} c$) to a constant $c$ if and only if for all $\varepsilon > 0$ \( \sum_{n=1}^{\infty} P(|X_n - c| > \varepsilon) < \infty \). Complete convergence implies convergence in probability.

The main result of this section is the following theorem describing complete convergence for $P(V_n)$. 

**Theorem 5.1.** For all $1 \leq p \leq d$ there exists constants $\mu_p^{d,p}$ and $\mu_Y^{d,p}$ such that

$$P(V_n) \overset{c.c.}{\to} \mu_p^{d,p},$$

(16)

and,

$$S_{\lfloor n/2 \rfloor}(V_n) \overset{c.c.}{\to} \mu_Y^{d,p}.$$  

(17)

We will prove this theorem later in this section.

### 5.1. Lipschitz continuity of $P(V)$ and $S_k(V)$

For the results in this section we need an extension of Lemma 2.6 given in [31]. From [2] it is known that that the maximum degree $d(v)$ of any vertex $v$ in the minimum spanning tree is bounded by a constant $c(d)$ depending only on the dimension $d$. E.g. for $d = 2$ it is known that $c(d) = 6$.

The following lemmas show that if two point sets $V$ and $W$ are 'close' then $|P(V) - P(W)|$ and $|S_k(V) - S_k(W)|$ are bounded, showing Lipschitz continuity for the $P(\cdot)$ and $S_k(\cdot)$-function, $k = 1, \ldots, |V| - 1$. 
Figure 1: Situation sketch for the proof of Lemma 5.1, showing $T_1, \ldots, T_6, V, V'$ and $W$. The normal edges are part of $T(V \cup W)$, the dashed edges are part of $T(W)$ but not of $T(V \cup W)$, $V'$ is obtained from $V$ by removing $u$, $T_5$, and $T_6$.

**Lemma 5.1.** Let $V$ and $W$ be sets of points in $\mathbb{R}^d$. Moreover, suppose that the edges in $T(W)$ and $T(V \cup W)$ have weight at most $C$. Then (a) $P(W) \leq P(V \cup W) + 2c(d)C|V \setminus W|$, and, (b) $S_k(W) \leq S_k(V \cup W) + (c(d) - 1)C|V \setminus W|$, for all $k = 1, \ldots, |V| - 1$.

**Proof.** We prove this lemma by induction on $|V \setminus W|$. The case where $|V \setminus W| = 0$ is obvious. Suppose the lemma holds for all sets $V, W$ with $|V \setminus W| < p_0$, $p_0 > 0$. To prove the inequalities for $|V \setminus W| = p_0$, consider any vertex $u \in V \setminus W$, and let $d$ be the degree of $u$ in $T(V \cup W)$. By deletion of $u$, the minimum spanning tree $T(V \cup W)$ is subdivided into a forest $F$ consisting of $d$ components, (see Figure 1 for a sketch of the situation). Let $\ell$ denote the number of components that contain vertices of $W$, $1 \leq \ell \leq d$ and number the components in such a way that $T_1, \ldots, T_\ell$ contain vertices of $W$, and $T_{\ell+1}, \ldots, T_d$ contain only vertices of $V$. So $W \subseteq T_1 \cup \ldots \cup T_\ell$. Define $V'$ as the set of vertices of $V$ which are part of one of the trees $T_1, \ldots, T_\ell$ (so
\(V' = V \setminus \{u, T_{\ell+1}, \ldots, T_d\}\). Because \(u \notin V'\) we get \(|V' \setminus W| \leq |V \setminus W| - 1\). First, we show that all edges in the minimum spanning tree on \(G(V' \cup W)\) have weight at most \(C\).

By Lemma 2.2, the minimum spanning tree \(T'\) on \(G((V \cup W) \setminus \{u\})\) contains \(T_1, \ldots, T_d\). As, by hypothesis, the maximum edge weight of \(T(V \cup W)\) is at most \(C\), this holds for all edges in \(T_i\), \(i = 1, \ldots, \ell\). The other edges in \(T'\) are between different components \(T_i\) and \(T_j\) with \(1 \leq i < j \leq \ell\). Note that these edges are the only edges of \(T'\) that are not part of \(T(V \cup W)\). In order to bound their weight, note that \(T(W)\) on \(W\) has maximum edge weight at most \(C\). From this tree we select \(\ell - 1\) edges, such that the addition of these edges to \(T_1, \ldots, T_\ell\) is a tree on \(G(V' \cup W)\) with maximum edge weight at most \(C\). By the fact that the minimum spanning tree minimizes the maximum edge weight, the spanning tree \(T'\) also has maximum edge weight at most \(C\).

Next, we bound \(P(V' \cup W)\) in terms of \(P(V \cup W)\). As each edge between different \(T_i\) and \(T_j\) contributes at most \(2C\) to the power assignment \(P(V' \cup W)\), we get \(P(V' \cup W) \leq P(V \cup W) + 2(\ell - 1)C\). Since for \(V'\) and \(W\) we have \(|V' \setminus W| < p_0\), we get by the induction hypothesis that:

\[
P(W) \leq P(V' \cup W) + 2(c(d) - 1)C|V' \setminus W|
\]
\[
\leq P(V \cup W) + 2(\ell - 1)C + 2(c(d) - 1)C|V' \setminus W|
\]
\[
\leq P(V \cup W) + 2(c(d) - 1)C(1 + |V' \setminus W|)
\]
\[
\leq P(V \cup W) + 2(c(d) - 1)C|V \setminus W|,
\]

where the second inequality follows from the fact that the number of components is bounded by the maximum degree of a vertex in a spanning tree, i.e., \(\ell \leq d \leq c(d)\). This finishes the proof for \(P(\cdot)\). The proof for \(S_k(\cdot)\) follows similarly. □

Note that in Lemma 5.1 we do not require a bound on the maximum edge weight of \(T(V)\), but only on that of \(T(W)\) and \(T(V \cup W)\). Moreover, in the proof we do not use a bound of \(C\) on the weight of the edges from \(T_i\) to \(T_j\), for \(1 \leq i \leq \ell\) and \(\ell + 1 \leq j \leq d\). The following example shows that all edges from \(T_i\) to \(T_j\) could have length exceeding \(C\). Assume that there is only one edge between \(V\) and \(W\) with weight at most \(C\). If \(u\) would be chosen incident to this edge, then all edges from \(T_i\) to \(T_j\), for \(1 \leq i \leq \ell\) and \(\ell + 1 \leq j \leq k\) have length exceeding \(C\).
Lemma 5.2. Let $V$ and $W$ be sets of points in $\mathbb{R}^d$. Suppose that $T(V)$, $T(W)$ and $T(V \cup W)$ are unique minimum spanning trees on $G(V)$, $G(W)$, and $G(V \cup W)$ with maximum edge weight at most $C$.

1. $P(V \cup W) \leq P(V) + (c(d) + 1)C|W\setminus V|$

2. If $|V| = |W|$ then $|P(V) - P(W)| \leq (3c(d) - 1)C|W\setminus V|$.

Proof. First note that for $V = W$ the lemma is obviously true. Thus, we assume $V \neq W$. Now, the proof consists of two steps. Since all edges in $T(V \cup W)$ have weight at most $C$, there is at least one edge $e = \{x, y\}$ with $x \in V$, $y \in W\setminus V$ and $c(e) \leq C$. Let $H = V \cup \{y\}$ and consider a minimum spanning tree $T(H)$ on $H$. Let $d = d(x)$ denote the degree of $x$ in $T(H)$. By Lemma 2.2, and the MST Algorithm, it follows that $P(H) \leq P(V) + (d + 1)C \leq P(V) + (c(d) + 1)C$. Now the first statement follows by iteration.

If, in addition, $|V| = |W|$, then by Lemma 5.1 we have that:

$$P(W) \leq P(V \cup W) + 2(c(d) - 1)C|V\setminus W|.$$ (18)

Combining the bounds, it follows that

$$
\begin{align*}
P(V) & \leq P(V \cup W) + 2(c(d) - 1)C|V\setminus W| \\
& \leq P(V) + (c(d) + 1)C|W\setminus V| + 2(c(d) - 1)C|V\setminus W| \\
& = P(V) + (3c(d) - 1)C|W\setminus V|,
\end{align*}
$$

using that $|V\setminus W| = |W\setminus V|$. By reversing the roles of $V$ and $W$, a bound follows for $P(W)$, completing the proof. $\square$

The proofs for $S_k(\cdot)$ are entirely similar, though smaller constants are possible:

Lemma 5.3. Let $V$ and $W$ be sets of points in the unit square. Suppose that $T(V)$, $T(W)$ and $T(V \cup W)$ are unique minimum spanning trees on $G(V)$, $G(W)$, and $G(V \cup W)$ with maximum edge weight at most $C$. Then, for $k = 1, \ldots, |V| - 1$,

1. $S_k(V \cup W) \leq S_k(V) + C|W\setminus V|$, and

2. If $|V| = |W|$ then $|S_k(V) - S_k(W)| \leq c(d)C|W\setminus V|$.
5.2. Complete convergence for $1 \leq p < d$

First we show complete convergence for $1 \leq p < d$:

**Theorem 5.2.** For all $1 \leq p < d$, there exist constants $\mu_{p,d}^P$ and $\mu_{p,d}^Y$ such that

$$\frac{P(V_n)}{n^{\frac{d-p}{2}}} \overset{c.c.}{\to} \mu_{p,d}^P,$$

and,

$$\frac{S_{\lfloor n/2 \rfloor}(V_n)}{n^{\frac{d-p}{2}}} \overset{c.c.}{\to} \mu_{p,d}^Y.$$

**Proof.** This follows from the fact that the functional $P(\cdot)$ and its associated boundary functional $P_B(\cdot)$ are smooth, i.e., $|P(V \cup W) - P(V)| = O(|W|^{\frac{d-p}{2}})$, for all point sets $V, W \subseteq [0,1]^d$. Moreover, $P(\cdot)$ is point-wise close to $P_B(\cdot)$ for $1 \leq p < d$. Both properties are inherited from the same properties for the MST functional. Now we can apply Theorem 4.1 and Corollary 6.4 from Yukich [30] to conclude complete convergence for both functionals.

The proof of the main theorem for $p = d$, where $p$ denotes the distance power gradient, and $d$ the dimension, goes in a number of steps, and first we show convergence in mean.

**Theorem 5.3.** For all $p = d \geq 2$, there exist constants $\mu_{p,d}^P$ and $\mu_{p,d}^Y$ such that

$$\lim_{n \to \infty} \mathbb{E}[P(V)] = \mu_{p,d}^P,$$

and

$$\lim_{n \to \infty} \mathbb{E}[S_{\lfloor n/2 \rfloor}(V)] = \mu_{p,d}^Y.$$

**Proof.** We only prove (21). The proof of (22) follows in a similar way. We first note that $\mathbb{E}[P(V)] \leq C$, for some constant $C > 0$. This immediately follows from the fact that for the minimum spanning tree $T(V)$, $\mathbb{E}[T(V)] \leq C$. If in addition, $P_B(V)$ is close in mean to $P(V)$ and $P(\cdot)$ is smooth in mean, then the theorem follows from [30] Theorem 4.5.

First, we show $P_B(V)$ is close in mean to $P(V)$, following the analogous proof for minimum spanning trees in [30] pp. 44 and 45. Let $P_B$ denote the power assignment associated to $T_B(V)$. We enumerate the components of $T_B$ by $T_1, \ldots, T_Q$ where $Q$ is random and where each $T_i$ represents a tree which is rooted to the boundary of the
unit cube. A minimum spanning tree can be obtained from $T_1, \ldots, T_Q$ by connecting $T_1, \ldots, T_Q$ by a set of edges $S$ such that $\sum_{e \in S} |e|^d$ is minimal. From the proof in [30], it follows that for any $\beta > 0$, there is a $C(\beta)$ such that $S$ can be chosen in such a way that with a probability of at least $1 - n^{-\beta}$ $S$ contains at most $C(\beta)n^{(d-1)/d}$ edges, with total weight at most $C(\beta)n^{-1/d}(\log n)^d$. By the fact that each added edge increases the weight of $P_B(V)$ with at most two times the edge weight, it follows that with a probability of at least $1 - n^{-\beta}$,

$$P_B(V) \leq P(V) \leq P_B(V) + 2C(\beta)n^{-1/2}(\log n)^d. \quad (23)$$

This shows that $P(V)$ is close in mean to $P_B(V)$, for all $d$.

Second, we have to show that $P(\cdot)$ is smooth in mean, so we must show that for any $\beta > 0$, there is a $C'(\beta)$ and $\gamma < 1/2$ such that for all $n \geq 1$ and $0 \leq k \leq n/2$ we have:

$$E|P(\{u_1, \ldots, u_n\}) - P(\{u_1, \ldots, u_{n+k}\})| \leq C'(\beta)kn^{-1+\gamma}. \quad (24)$$

The following statement implies (24). For any $\beta > 0$, there is a constant $C'$, depending only on $\beta$, so $C' = C'(\beta)$ such that with a probability of at least $1 - n^{-\beta}$

$$|P(\{u_1, \ldots, u_n\}) - P(\{u_1, \ldots, u_{n+k}\})| \leq C'(\beta)k(\log n/n), \quad (25)$$

To see (25), note that for any $\beta > 0$, there is a constant $C(\beta)$ such that with a probability of at least $1 - n^{-\beta}$ the edges in the MST on $V$ have length at most $C(\beta)(\log n/n)^{1/d}$. In this case, the maximum weight of an edge is $C(\beta)^d(\log n/n)$. To compensate for the fact, that we may also remove (at most $n/2$) points from $\{u_1, \ldots, u_n\}$, which increases $\log n/n$, we set $C'(\beta) = 2\beta C(\beta)^d$. It follows from the first statement of Lemma 5.2 that $P(\{u_1, \ldots, u_{n+k}\}) \leq P(\{u_1, \ldots, u_n\}) + (c(d) + 1)kC'(\beta)(\log n/n)$. To see $P(\{u_1, \ldots, u_n\}) \leq P(\{u_1, \ldots, u_{n+k}\}) + (2c(d) - 1)kC'(\beta)(\log n/n)$, apply Lemma 5.1 with $V = \{u_{n+1}, \ldots, u_{n+k}\}$ and $W = \{u_1, \ldots, u_n\}$. By similar reasoning we deal with the case $|P(\{u_1, \ldots, u_n\}) - P(\{u_1, \ldots, u_{n-k}\})|$. □

5.3. Concentration result

In order to show Theorem 5.2, we provide some further definitions. Throughout, $\mu$ denotes the volume (in this case ‘Euclidean area’) measure in $\mathbb{R}^d$ and $\mu^n$ denotes the $n$-fold product measure on $([0,1]^d)^n$. (Effectively, $\mu^n$ is the volume in $[0,1]^{dn}$.)
Let \( x = (x_1, \ldots, x_n) \) be an \( n \)-tuple in \(((0, 1]^d)^n\). Recall that the Hamming distance \( H \) on \(((0, 1]^d)^n\) measures the distance between \( x \) and \( y \) by the number of coordinates in which \( x \) and \( y \) disagree: \( H(x, y) = \text{card}\{i : x_i \neq y_i\} \). Note that with \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \), we can associate the unordered sets \( x' = \{x_1, \ldots, x_n\} \) and \( y' = \{y_1, \ldots, y_n\} \). Now with \( l(x', y') = |x' \setminus y'| = |\{x \in x' | x \notin y'\}| = |x'| - |x' \cap y'| \), we immediately have: \( H(x, y) \geq l(x', y') \).

For all \( t > 0 \) the \( t \)-enlargement of a set \( A \subseteq ([0, 1]^d)^n \) is the set of tuples with small Hamming distance to \( A \), defined by

\[
A_t := \{x \in ([0, 1]^d)^n : \exists y \in A \text{ such that } H(x, y) \leq t\}.
\]

The following theorem is shown by Talagrand [28], Lemma 5.1.

**Theorem 5.4.** ([28].) Let \( A \subseteq ([0, 1]^d)^n \), and let \( A^c_t \) denote the complement of \( A_t \). Then for all \( t > 0 \),

\[
\mu^n(A_t^c) \leq \frac{1}{\mu^n(A)} e^{-\frac{t^2}{n}}.
\]

To show Theorem 5.2, we need some additional results. First we show that, for \( C \) large enough, the following set of grid points closely approximates an arbitrary set of \( n \) points. Here \( \pi_d \) denotes the volume of a hyperball of unit radius in \( \mathbb{R}^d \).

**Lemma 5.4.** Let \( \{g_i\}_{i=1}^n \) denote a collection of grid points in \([0, 1]^d\) (i.e., a lattice in \(((0, 1]^d)^n\) with \( n^{-1/d} \) spacing) and, for a fixed \( C > 0 \), let \( D \) denote the subset of \([0, 1]^d\) with the property that for each grid point there is some point in the set that is ‘close’:

\[
D \equiv \{x = (x_1, \ldots, x_n) \in ([0, 1]^d)^n : \max_{1 \leq j \leq n} \text{dist}(g_j, \{x_i\}_{i=1}^n) \leq C(\log n/n)^{1/d}\}.
\]

Then

\[
\mu^n(D^c) \leq n^{1-\pi_d 2^{-d} C^d}.
\]

**Proof.** Consider \( D^c \) and note that \( D^c \subset \bigcup_{j=1}^n D_j^c \), where \( D_j^c \) is defined as: \( D_j^c = \{x \in ([0, 1]^d)^n : \text{dist}(g_i, \{x_j\}_{j=1}^n) > C(\log n/n)^{1/d}\} \). Clearly, \( \mu^n(D^c) \leq \sum_{i=1}^n \mu^n(D_i^c) \).

We also have, for any \( i = 1, \ldots, n \), with

\[
\mu^n(D_i^c) \leq \left(1 - 2^{-d} \pi_d C^d (\log n/n)\right)^n.
\]

To see this, note that with \( D_{i,j}^c = \{x \in ([0, 1]^d)^n : \text{dist}(g_i, x_j) > C(\log n/n)^{1/d}\} \), we have \( D_i^c = \bigcap_{j=1}^n D_{i,j}^c \), and \( \mu(D_{i,j}^c) = 1 - \pi_d C^d (\log n/n) \), if the hyperball centered at \( g_i \)
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with radius \((C \log n/n)^{1/d}\) is fully contained in \([0, 1]^d\). As at least a fraction of \(2^{-d}\) of this hyperball is contained in \([0, 1]^d\), it follows that, \(\mu(D_{c,i,j}^c) \leq 1 - 2^{-d} \pi d C^d (\log n/n)\). As \(1 - x \leq e^{-x}\) for \(x \geq 0\), we have, \(\mu^n(D_c^c) \leq (1 - 2^{-d} \pi d C^d \log n/n) n \leq e^{-2^{-d} \pi d C^d \log n} = n^{-2^{-d} \pi d C^d}\). Hence \(\mu^n(D_c^c) \leq \sum_{i=1}^n \mu^n(D_{c,i}^c) \leq n^{1 - \pi d 2^{-d} C^d}\). □

Let us first provide an intuitive explanation for the proof of Theorem 5.2. For fixed \(n\), with \(V = \{U_1, \ldots, U_n\}\) we let \(A_n\) denote all sets of \(n\) vertices on \([0, 1]^d\), for which \(P(V)\) exceeds the median weight \(M_P(n)\), and define the \(t\)-enlargement \(A_t\) of \(A\), where \(t = t(n)\), and \(A_t\) depends on a constant \(C\), in such a way that for any \(\beta > 0\) there is a \(C(\beta)\) such that \(\mu(A_t) \geq 1 - n^{-\beta}\). The definition of \(A_t\) and some technical constructions ensures that if \(y \in A_t\) then \(P(y)\) is ‘close’ to \(M_P(n)\). It follows that with a probability of at least \(1 - n^{-\beta}\) any set of vertices has \(P(V)\) ‘close’ to \(M_P(n)\).

In order to bring us in a position to apply Lemma 5.2 we need to make sure that if \(y \in A_t\) then there is some \(x \in A\) so that \(x\) and \(y\) are ‘close’ and so that the maximum weight of the minimum spanning tree edges on \(x\) resp. \(y\) is ‘small’. That is the role of the set \(D\) of grid points defined in Lemma 5.4. They provide a point of reference: if \(x\) and \(y\) are both close to \(D\) then \(x\) must be close to \(y\). The role of the set \(B\) (the minimum spanning trees with ‘short’ edge lengths) is to make sure that we are only dealing with minimum spanning trees that do not violate the assumptions for Lemma 5.2. We show:

**Theorem 5.5.** For any \(d \geq 2\), let \(G(V)\) be as described before (so with \(p = d\)), then

\[
\lim_{n \to \infty} \mathbb{E}[|P(V) - M_P(n)|] = 0, \quad \text{and} \quad \lim_{n \to \infty} |P(V) - M_P(n)| \xrightarrow{c.c.} 0.
\]

where \(M_P(n)\) denotes a median of \(P(V)\). With \(k = k(n) = \lfloor n/2 \rfloor\) it holds that,

\[
\lim_{n \to \infty} \mathbb{E}[|S_k(V) - S(n)|] = 0, \quad \text{and} \quad \lim_{n \to \infty} |S_k(V) - S(n)| \xrightarrow{c.c.} 0.
\]

where \(M_S(n)\) denotes a median of \(S_k(V)\).

We will only prove statements (26) and (27). The proof of (28) and (29) follows the same lines. For ease of notation we write \(M(n)\) instead of the \(M_P(n)\), as defined in Theorem 5.5. First we show,
Lemma 5.5. For any \( d \geq 2 \) and for any \( \epsilon, \beta > 0 \):
\[
\mathbb{P}(|P(V) - M(n)| > \epsilon) \leq O(n^{-\beta}) + 3\exp\left(-\frac{n}{(\log n)^2 \beta^2 D_0^2}\right),
\]
(30)
where \( D_0 = (3c(d) - 1)C(\beta)^d \), where \( C(\beta) \) is chosen so that with a probability of \( 1 - n^{-\beta} \) the edges in the MST on \( \{x_i\}_{i=1}^n \) have length at most \( C(\beta)(\log n/n)^{1/d} \).

Proof. To see (33), fix \( \epsilon > 0 \) and \( \beta > 0 \) and let \( A \subseteq ([0,1]^d)^n \) consist of those \( n \)-tuples \( V := (u_1, \ldots, u_n) \in ([0,1]^d)^n \) for which
\[
P(V) \geq M(n).
\]

By definition of \( M(n) \), \( \mu^n(A) \geq 1/2 \). Choose \( C(\beta), \) so that \( 2^{-d} \pi_d C(\beta)^d > \beta \), and so that with a probability of at least \( 1 - n^{-\beta} \), the edges in the MST on \( \{x_i\}_{i=1}^n \) have length at most \( C(\beta)(\log n/n)^{1/d} \). Let \( B \subset ([0,1]^d)^n \) consist of all \( n \)-tuples \( x = (x_1, \ldots, x_n) \) such that the edges in the MST on \( \{x_i\}_{i=1}^n \) have length at most \( C(\beta)(\log n/n)^{1/d} \). By further increasing \( C(\beta) \) we can ensure that with \( D \) as in Lemma 5.4, it follows that
\[
\mu^n(B) \geq 1 - n^{-\beta}, \text{ and } \mu^n(D) \geq 1 - n^{-\beta}.
\]

Thus, we easily have \( \mu^n(A \cap B \cap D) \geq 1/3 \).

For \( n \) large enough, it follows by Talagrand’s theorem Theorem 5.4 (see [28], Lemma 5.1), that if we define \( t = t(n) = D_0^{-1}\epsilon (n/\log n) \) with \( D_0 = (3c(d) - 1)C(\beta)^d \), the volume of the enlarged set \( \mu^n(A \cap B \cap D)_t \leq O(n^{-\beta}) : \)
\[
\mu^n((A \cap B \cap D)_t) \leq 3\exp\left(-\frac{n}{(\log n)^2 \beta^2 D_0^2}\right).
\]
(31)

Now define \( E := (B \cap D) \cap (A \cap B \cap D)_t \), so \( E \) is the set of points ‘close’ to a grid point with ‘short’ edges in the minimum spanning tree and not deviating ‘too’ much from \( A \cap B \cap D \). Note that
\[
\mu^n(E^c) \leq \mu^n(B^c) + \mu^n(D^c) + \mu^n((A \cap B \cap D)_t^c) = O(n^{-\beta}) + 3\exp(-f_d(n))
\]
where \( f_d(n) \) is shorthand notation for the expression in (31). We now show that if \( x \in E \) then \( |P(x) - M(n)| \) is bounded. Suppose \( x := (x_1, \ldots, x_n) \in E \), then \( x \in (A \cap B \cap D)_t \) and so there is a point \( y := y(x) = (y_1, \ldots, y_n) \in A \cap B \cap D \) such that \( H(x,y) \leq t \).

Since \( x \) and \( y \) are both in \( B \), the edges in the graph of the minimal spanning tree on \( x \) and \( y \) have length bounded by \( C(\beta)(\log n/n)^{1/d} \). Since \( x \) and \( y \) are both in \( D \), \( y \) is close to \( x \) in the sense that \( \max_{1 \leq i \leq n} \text{dist}(x_i, \{y_j\}_{j=1}^n) \leq C(\beta)((\log n/n)^{1/d}. \)
and \(\max_{1 \leq i \leq n} \text{dist}(y_i, \{x_j\}_{j=1}^n) \leq C(\beta)(\log n/n)^{1/d}\). By Lemma 5.2, the fact that \(H(x, y) \geq l(x', y')\), and weights are the \(d\)-th powers of the distances, our definition of \(t(n)\) implies:

\[
|P(y) - P(x)| \leq (3c(d) - 1)t(n)C(\beta)^d \left(\frac{\log n}{n}\right) = \epsilon.
\]

Therefore, for all \(x \in E\) and \(y = y(x) \in A \cap B \cap D\) as above, we have

\[
P(x) \geq P(y) - |P(x) - P(y)| \geq M(n) - \epsilon.
\]

Thus it follows for general \(V = (u_1, \ldots, u_n) \in ([0, 1]^\beta)^n\) that

\[
P(P(V) < M(n) - \epsilon) \leq \mu^n(E^c) \leq O(n^{-\beta}) + 3\exp(-f_d(n)).
\]

and

\[
P(P(V) > M(n) + \epsilon) \leq O(n^{-\beta}) + 3\exp(-f_d(n))
\]

and so

\[
P(|P(V) - M(n)| > \epsilon) \leq O(n^{-\beta}) + 6\exp(-f_d(n)). \tag{32}
\]

This shows (33). □

Note that Lemma 5.5 immediately implies (27),

\[
|P(V) - M(n)| \xrightarrow{c.c.} 0.
\]

**Proof.** Consider (33):

\[
P(|P(V) - M(n)| > \epsilon) \leq O(n^{-\beta}) + 3\exp\left(-\frac{n}{(\log n)^2 D_0^2} \epsilon^2\right), \tag{33}
\]

So, for all \(\epsilon > 0\) we have

\[
\sum_{n=2}^\infty P(|P(V) - M(n)| > \epsilon) \leq \sum_{n=2}^\infty O(n^{-\beta}) + 3 \sum_{n=2}^\infty \exp\left(-\frac{n}{(\log n)^2 D_0^2} \epsilon^2\right) = O(1). \tag{34}
\]

Thus we have complete convergence to zero. □

**Proof of Theorem 5.5.** We proceed by showing (26). For \(x = \{x_1, \ldots, x_n\}\), consider \(|P(x) - M(n)|\). First note that \(|P(x) - M(n)| \leq C\) for an appropriate constant \(C > 0\) and all \(x\). To see this, observe that, \(0 \leq P(x) \leq C\), for some constant \(C\), depending on \(d\). For \(M(n)\) clearly the same is true. Combining the identity \(\mathbb{E}[|P(x) - M(n)|] = \int_0^C P(|P(x) - M(n)| > t)dt\) with the estimate (32) implies

\[
\mathbb{E}[|P(x) - M(n)|] \leq O(n^{-\beta}), \tag{35}
\]
and as \( \beta > 1 \), the righthand side of (35) tends to 0 when \( n \to \infty \). This shows (26). \( \square \)

Finally, we are in the position to give the proof of the main theorem of this section.

**Proof of Theorem 5.2.** By Theorem 5.3 \( \lim_{n \to \infty} E[P(V)] = \mu_d^T \) and by (26) \( \lim_{n \to \infty} ||P(V) - M(n)|| = 0 \), the triangle inequality implies (19). Now (20) follows in a similar way from (28) and (29). \( \square \)

**Corollary 5.1.** Let \( \mu_d^T, \mu_d^P, \mu_d^Y \) be so that \( T(V) \overset{c.c.}\to \mu_d^T, P(V) \overset{c.c.}\to \mu_d^P, \) and \( S_{\lfloor n/2 \rfloor}(V) \overset{c.c.}\to \mu_d^Y \). Then,

\[
\lim_{n \to \infty} E\left[ \frac{P(V)}{T(V)} \right] = \frac{\mu_d^P}{\mu_d^T} \leq 2\frac{\mu_d^Y}{\mu_d^T} \tag{36}
\]

**Proof.** By Lemma 2.1 we find \( \mu_d^P \leq 2\mu_d^Y \). The fact that \( \frac{P(V)}{T(V)} \) is bounded from above and below implies the statement. \( \square \)

### 6. Conclusions and further research

This paper presents an average case analysis of the minimum spanning tree heuristic for the range assignment problem on a graph with power weighted edges. The worst-case approximation ratio of this heuristic is 2. Our analysis yields the following results:

(a) In the one-dimensional case \( (d = 1) \), where the weights of the edges are 1 with probability \( p \) and 0 otherwise, the expected approximation ratio for \( n \to \infty \) tends to \( 2 - p \).

(b) In the one-dimensional case, with the distance between neighboring vertices drawn from a uniform \([0, 1]\)-distribution, the expected approximation ratio is bounded from above by \( 2 - 2/(p + 2) \) where \( p \) denotes the distance power gradient.

(c) For the complete graphs with uniformly distributed edge weights, the expected approximation ratio is asymptotically bounded from above by \( 2 - 1/2\zeta(3) \approx 1.58... \).

(d) For the case with \( 1 \leq p \leq d \), the weight of the power assignment divided by \( n^{(p-d)/d} \) converges completely.

In order to show this, we use extensions of results of Yukich [31], that require general results on minimum spanning trees in graphs that are of interest by itself.

Concerning the MST heuristic, it would be interesting to investigate heuristics as presented in [3]. Further extension of this type of results to power assignments resulting in general \( k \)-connected graphs might also be possible.
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References


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