Fitting world-wide web request traces with the EM-algorithm

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Abstract

In recent years, various researchers have shown that network traffic that is due to world-wide web transfers shows characteristics of self-similarity and it has been argued that this can be explained by the heavy-tailedness of many of the involved distributions. Considering these facts, developing methods that are able to handle self-similarity and heavy-tailedness is of great importance for network capacity planning purposes.

However, heavy-tailed distributions cannot be used so easily for analytical or numerical evaluation studies. To overcome this problem, in this paper, we approximate the empirical distributions by analytically more tractable, that is, hyper-exponential distributions. For that purpose, we present a new fitting algorithm based on the expectation-maximisation and show it to perform well both for pure traffic statistics as well as in queuing studies.

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1. Introduction

Over the last decade, extensive traffic measurements have shown the presence of properties such as self-similarity, fractality and long-range dependency in network traffic. The seminal paper by Leland et al. [10] showed self-similarity in Ethernet traffic; later, similar effects were shown to exist in wide area network traffic, signalling traffic, and in multimedia and video traffic. Crovella and Bestavros [11] have shown that network traffic that is due to WWW transfers can show characteristics that are consistent with self-similarity and argue that this can be explained by the heavy-tailedness of many of the involved distributions. Since the majority of all Internet traffic is due to HTTP transfers [6], understanding the nature of such traffic is crucial for the construction and evaluation of future Internet applications, as it has been shown that ignoring the above properties (that is, self-similarity, heavy-tailedness) in the analysis of

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Various efforts have been pursued to develop appropriate traffic models to evaluate the performance of systems under self-similar traffic. Often, the first step is to construct heavy-tailed distributions (HTDs) to approximate the involved empirical distributions in the measurement data. However, “classical” HTDs cannot be used so easily for analytical or numerical evaluation studies, since the latter often rely on the use of Poisson or other “Markovian” distributions. To overcome this problem, various approaches to approximate HTDs by analytically more tractable distributions have been proposed [2,4,5,15].

Of particular interest is the use of hyper-exponential distributions (HEDs) to approximate HTDs, since these distributions are very well understood and well suited for analytical and numerical performance studies. The use of HEDs for this purpose has been proposed by Feldmann and Whitt [2] (denoted here as the “FW-approach”). Although the FW-approach is fast, it requires an explicit representation of an HTD to fit to. Such an explicit HTD can for instance be obtained by fitting a Weibull or a Pareto distribution to the measurement data using the maximum-likelihood (ML) method. However, as we will see below, often the measurements to be fitted do not suit a Weibull or a Pareto distribution well, so that the thus-obtained HED does not describe the measurements well. The FW-approach is illustrated in the (dark) grey box in Fig. 1.

To avoid the use of an intermediate HTD, we have decided to directly fit an HED to the measured data via both moment matching (see Appendix B) and the expectation-maximisation (EM) algorithm (see Appendix C). This is described in detail in the paper; see also the hatched box in Fig. 1.

To validate our new fitting approach, we use a large trace from the RWTH world-wide web proxy server as well as a well-known NASA web server request-log and fit the requested object-size distribution. We both study the fit itself, as well as performance results obtained when using the fitted distributions in a matrix-geometric analysis of an M/G/1 queue for different utilisations (also in Fig. 1; the right-most comparison).

This paper is further organised as follows. We will give some background on HTDs and HEDs in Section 2. Then, we summarise the FW-approach in Section 3. Our new fitting approach is presented in Section 4 and validated in Section 5. The paper is concluded in Section 6 with some final remarks. Three appendices on statistical fitting procedures are included to keep the paper self-contained.

![Fig. 1. Graphical representation of the two fitting procedures.](image-url)
Table 1
Characteristics of the Pareto and the Weibull distribution

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Density $f(x)$</th>
<th>Expectation</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pareto</td>
<td>$akx^{-(a+1)}$</td>
<td>$\frac{ak}{a-1}$ for $a &gt; 1$</td>
<td>$\frac{ak^2}{2(a-1)^2}$ for $a &gt; 2$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$\frac{b}{ax}e^{-\frac{x}{b}}$ for $a &gt; 0$ and $b &gt; 0$</td>
<td>$\frac{a}{b}\Gamma(1/b)$</td>
<td>$\frac{ab}{b^2}\Gamma(2/b) - \left[\frac{\Gamma(1/b)^2}{\Gamma(2/b)}\right]$</td>
</tr>
</tbody>
</table>

2. HTDs and HEDs

2.1. Heavy-tailed distributions

Self-similarity in network traffic has been explained by the fact that many of the involved distributions, e.g., of file sizes, are heavy-tailed [11]. In an HTD, the complementary cumulative distribution function $F$ decays more slowly than exponentially, that is, $e^{-\gamma t} F(x) \to \infty$ as $x \to \infty$ for all $\gamma > 0$. For a random variable $X$, distributed according to some HTD, we typically have

$$\text{Pr}\{X > x\} \sim x^{-\alpha}, \quad \text{for } x \to \infty, \quad 0 < \alpha < 2.$$  \hfill (1)

Note that “$x \to \infty$” should be read as “for very large $x$” in case of measurements.

The degree of the heavy-tailedness is given by the shape parameter $\alpha$ which can be determined by plotting the complementary cumulative distribution $F(x) = 1 - F(x) = \text{Pr}\{X > x\}$ on a log–log scale. The slope of the plot, found, for instance, via a linear regression, then gives the value of $\alpha$.

In Table 1, we list some characteristics of two well-known HTDs, the Pareto and the Weibull distribution (in case the stated conditions are not met, the expectation and/or variance do not exist) [9].

2.2. Hyper-exponential distributions

An HED can be interpreted as a probabilistic choice between $I$ exponential distributions (see Fig. 2) [7] and is an example of a so-called phase-type distribution [13]. With (initial) probability $c_i$ the $i$th negative exponential distribution (with rate $\lambda_i$ and mean $1/\lambda_i$) is chosen. Such an $I$-phase HED has distribution function

$$F(x) = 1 - \sum_{i=1}^{I} c_i e^{-\lambda_i x}, \quad x \geq 0$$  \hfill (2)

Fig. 2. Graphical representation of an $I$-phase HED.
and density function

\[ f(x) = \sum_{i=1}^{I} c_i \lambda_i e^{-\lambda_i x}, \quad x \geq 0. \]  

(3)

The \( j \)th moment is given by

\[ E[X^j] = j! \sum_{i=1}^{I} c_i \lambda_i^j. \]  

(4)

3. Approximation of HTDs with HEDs

The FW-approach [2] comprises an efficient elegant method to approximate an HTD with an HED. The method is often applied due to its simplicity and efficiency [3,4]. We first summarise the approach before we discuss our experience with it.

3.1. The FW-approach

In the FW-approach, it is assumed that an HTD is given in an explicit form. How this HTD is obtained from, for instance, measurement data, is not a part of it. Provided that an explicit representation of the HTD \( F(x) \) is available, an \( I \)-phase HED distribution of the form given in (2) is found. Note that, for \( I \to \infty \), one can represent any distribution, with squared coefficient of variation at least 1, with completely monotone probability density function arbitrary close by hyper-exponentials [2]. However, it has been shown [2] that with values of \( I \) up to 20, HTDs can approximate Weibull and Pareto distributions for large ranges of \( x \).

For a given HTD \( F(x) \) and an a priori fixed number of phases \( I \) the FW-approach operates as follows:

1. Choose quantiles \( 0 < q_1 < q_{I-1} < \cdots < q_I \) with sufficiently large ratio \( q_i/q_{i+1} \), e.g., \( q_i/q_{i+1} \approx 10 \) (for \( i = 1, \ldots, I - 1 \)). Furthermore, let \( b \) be such that \( 1 < b < q_i/q_{i+1} \) for all \( i \).
2. In \( I \) steps, the parameters for the phases in the HED are computed. We start with setting \( i := 1 \) and \( F_c^1(q_i) = F^1_i(x) = 1 - F(x) \).
3. In the \( i \)th phase, we compute \( c_i \) and \( \lambda_i \) by solving the equations:

\[ c_i e^{-\lambda_i q_i} = F_c^i(q_i), \quad c_i e^{-\lambda_i bq_i} = F_c^i(bq_i), \]

yielding (explicitly)

\[ \lambda_i = \frac{1}{(b - 1)q_i} \ln \left( \frac{F_c^i(q_i)}{F_c^i(bq_i)} \right), \quad c_i = F_c^i(q_i) e^{\lambda_i q_i}. \]

(4)

Step 3 is repeated for \( i = 2, \ldots, I - 1 \), where

\[ F_c^i(q_i) = F_c^{i-1}(q_i) - c_{i-1} e^{-\lambda_{i-1} q_i}, \quad F_c^i(bq_i) = F_c^{i-1}(bq_i) - c_{i-1} e^{-\lambda_{i-1} bq_i}. \]
Finally, for the last phase \( I \) we find
\[ c_I := 1 - \sum_{j=1}^{I-1} c_j \]
and \( \lambda_I \) follows from \( c_I e^{-\lambda_I q_I} = F_I(q_I) \).

The complexity of the algorithm is \( O(I) \) where each step consists of solving a system of two, in fact, linear equations. However, since the algorithm cannot be applied directly to measurement data, the costs of an algorithm, like the ML algorithm [1] (cf. Appendix A), or a moment matching algorithm (cf. Appendix B), to fit the measurements to an explicit HTD must be considered as well.

### 3.2. Application and validation

When applying the FW-approach to find object-size distributions from the log-files used in our case studies (for a detailed description of the traces and the statistical parameters, see Section 5), we found that the typically employed HTDs, like Pareto and Weibull, do not describe the object-size distributions well. Both distributions fit the tail of empirical measurement distribution well, but fail to fit the head properly. For example, a Weibull distribution whose first and second moment have been fitted to the data, results in a median that is half the median found in the data (the median is located in the head, see Tables 3 and 4 in Section 5).

Hence, even when the FW-approach does give a good fit with respect to a given HTD, if the provided HTD does not describe the data well, then the finally fitted HED does not describe the measurements well, either.

Feldmann and Whitt [2] point out that it might be possible to extend their approach so it can be directly applied to measurement data. They also warned that the algorithm, at least without extension, is not designed to directly treat data but might well be applied after some initial smoothing of the data. We have performed a number of experiments in this direction. In fact, these experiments have shown that the smoothing is absolute necessary, since otherwise the algorithm is too sensitive to the location of the quantiles \( q_i \). Furthermore, the quality of the approximation heavily depends on the quality of the smoothing. Simple smoothing methods based on linear or square interpolation did not yield satisfactory results.

The above observations have led us to develop another direct fitting approach, as will be outlined below.

### 4. A direct fitting approach for HEDs

The EM-algorithm is a well-known algorithm to fit measurements to distributions [8,18,20,21]; a detailed description can be found in Appendix C. The EM-algorithm operates in an iterative fashion and does require neither an intermediate HTD nor any heuristics. Below, we outline the method in general, and then specialise it to the case where the distribution function to fit to is an HED.
4.1. General approach

Given measurement data \( x_1, \ldots, x_N \), we search the parameters \( c = (c_1, \ldots, c_I) \) and \( \theta = (\theta_1, \ldots, \theta_I) \) of the density function
\[
p(x|c, \theta) = \sum_{i=1}^{I} c_i p(x|\theta_i),
\]
so that it “best” fits the density of the measurement data. The density in (5) is a convex combination of basic density functions \( p(x|\theta_i) \) parameterised by \( \theta_i \) with weights \( c_i \geq 0 \) and \( \sum_{i=1}^{I} c_i = 1 \). For instance, for Weibull distributions as basic distributions \( \theta_i = (a_i, b_i) \), and for exponential distributions we would have \( \theta_i = (\lambda_i) \).

Now, let \( \alpha = (c, \theta) \) and \( \hat{\alpha} = (\hat{c}, \hat{\theta}) \) be two sets of parameters for the density \( p \). The EM-algorithm defines a new density function
\[
\delta(i|x_n, \theta_i) = \frac{c_i p(x_n|\theta_i)}{p(x_n|\alpha)},
\]
as well as the function \( Q \), which provides a quality criterion for \( \alpha \) and \( \hat{\alpha} \), that is, it says how much better the density function \( p(x|\hat{\alpha}) \) fits the measurement data than the density function \( p(x|\alpha) \):
\[
Q(\alpha, \hat{\alpha}) = N \sum_{n=1}^{N} I \sum_{i=1}^{I} \delta(i|x_n, \theta_i) \log \left( \frac{\hat{c}_i p(x_n|i, \hat{\lambda}_i)}{p(x_n|\alpha)} \right).
\]
The EM-algorithm proceeds iteratively: starting from an initial parameter set \( \alpha = (c, \theta) \), it computes a new parameter set \( \alpha' = (c', \theta') \) such that \( Q(\alpha, \hat{\alpha}) \) is maximised. This \( \alpha' \) is used as starting point for the next iteration. The algorithm stops when \( \alpha \approx \alpha' \) (see below). To find the next value \( \alpha' \), that is, to optimise \( Q \), one has to take derivatives to subsequently solve the equation system (possibly non-linear):
\[
\frac{\partial Q}{\partial \alpha} = 0 \Rightarrow \frac{\partial Q}{\partial \lambda_{i}} = 0, \ldots, \frac{\partial Q}{\partial \lambda_{i}} = 0.
\]
Using Lagrange multipliers (with auxiliary condition \( \sum_{i=1}^{I} c_i = 1 \)), the new weights \( c' \) are given by
\[
c'_i = \frac{1}{N} \sum_{n=1}^{N} c_i p(x_n|\theta_i).
\]
Substituting the function $p(x|\iota, \lambda')$ gives us

$$\sum_{n=1}^{N} \delta(i|X_n, \lambda_i) \frac{\partial}{\partial \lambda_i} \log (c_i' \lambda_i e^{-\lambda_i x_n}) = 0,$$

(11)

$$\Rightarrow \sum_{n=1}^{N} \delta(i|X_n, \lambda_i) \frac{\partial}{\partial \lambda_i} (\log c_i' + \log \lambda_i - \lambda_i x_n) = 0,$$

(12)

$$\Rightarrow \sum_{n=1}^{N} \delta(i|X_n, \lambda_i) \left( \frac{1}{\lambda_i} - x_n \right) = 0,$$

(13)

$$\Rightarrow \sum_{n=1}^{N} \delta(i|X_n, \lambda_i) = \sum_{n=1}^{N} \delta(i|X_n, \lambda_i)x_n.$$

(14)

From this we conclude that

$$\lambda_i' = \frac{\sum_{n=1}^{N} \delta(i|X_n, \lambda_i)}{\sum_{n=1}^{N} \delta(i|X_n, \lambda_i)x_n}.$$

(15)

The EM-algorithm specialised for HEDs, that is, with $p(x|\lambda_i) = \lambda_i e^{-\lambda_i x}$, now takes the following form:

1. Select an appropriate number of distributions $I$ to form the convex combination (4) and select initial values for $\lambda_i$ and $c_i$ $(i = 1, \ldots, I)$, as well as a positive required accuracy $\epsilon$.
2. Compute for $i = 1$ to $I$:
   (a) $p(i|X_n, \lambda_i) = c_i p(x_n|\lambda_i) / p(X_n|\alpha)$,
   (b) $c_i' = \frac{1}{N} \sum_{i=1}^{N} p(i|X_n, \lambda_i)$,
   (c) $\lambda_i' = \frac{\sum_{n=1}^{N} p(i|X_n, \lambda_i)}{\sum_{n=1}^{N} p(i|X_n, \lambda_i)x_n}$.
3. Return to step 2 with $c_i := c_i'$ and $\lambda_i := \lambda_i'$ until the difference between $c_i$ and $c_i'$ and the difference between $\lambda_i$ and $\lambda_i'$ (for all $i$) is smaller than the value $\epsilon$.

In the above algorithm, we have preset the number of phases $I$. In a different variant of the EM-algorithm, this number does not have to be preset, but is computed on-the-fly, thus yielding a number of phases that is large enough to describe the required HTD, yet as small as possible to keep the fitted HED small [8, 22]; see also some comments below.

4.3. Complexity and choice of initial values

The EM-algorithm is an iterative algorithm where the complexity of each iteration is $O(NI)$, with $N$ the number of measurements and $I$ the number of phases. A problem with the EM-algorithm is the fact that it is difficult to predict the number of iterations needed to reach a given precision of the result.
[20]. However, in our experiments, good results generally have been obtained within 5–10 iterations. Additionally, it should be noted that even for a case study with $N$ well over 17 million (see below) and $I = 6$, one iteration took less than 1 min on a standard personal computer (with a CPU clocked at 450 MHz and 512 MB main memory).

The number of required iterations is heavily influenced by the choice of the initial values. Here, we can use our knowledge about the shape of the distribution function to choose initial values that are near to the (expected) final results of the algorithm. We know, e.g., that data sizes in the world-wide web are described by HTDs, and that small documents are more popular than larger ones. That means the values for $c_i$ have to be decreasing according to the values of the data sizes, given by the inverses of $\lambda_i$. This results in the following guidelines for setting initial values for $c_i$ and $\lambda_i$ (for $i = 1, \ldots, I$):

$$c_i = \frac{0.9 \times 10^{-i}}{1 - 10^{-1}} \quad \text{and} \quad \lambda_i = 10^{-3-i} \quad \text{for } i = 1, \ldots, I.$$  \hspace{1cm} (16)

When the number of phases $I$ is not preset but computed on-the-fly (as mentioned at the end of Section 4.2), one starts with $I = 1$, $c_1 = 1.0$ and $\lambda_1 = N/(\sum_{n=1}^{\infty} x_i)$, that is, one starts with an expression according to ML-fitting (see Appendix A). The algorithm then chooses a new number of phases $I_{\text{new}} := 2I$, that is, the number of phases is doubled per iteration step. The new values of $c_{\text{new},i}$ and $\lambda_{\text{new},i}$ for $i = 1, \ldots, I_{\text{new}}$ are computed as a function of the previous values in the following way. Ranging $i$ over the odd values $i = 1, 3, \ldots, I_{\text{new}} - 1$, we set

$$c_{\text{new},i} = \frac{0.9 \times 10^{-i}}{1 - 10^{-1}} \quad \text{and} \quad c_{\text{new},i+1} = \frac{0.9 \times 10^{-i-1}}{1 - 10^{-1}}.$$  \hspace{1cm} (17)

It should be mentioned that other initial values for $c_i$ and $\lambda_i$ can be chosen, as well as other computational schemes when changing $I$. A further discussion of these issues, however, goes beyond the scope of the current paper.

5. Application and validation

In order to evaluate the presented fitting approaches, we applied them to two data traces (described in detail below). Using these, we made comparisons between:

1. first-order statistics of the obtained HEDs and those of the original data traces;
2. performance results for an $M|G|1$ queue where the original data traces and the fitted distributions are used as service-time distribution.

5.1. Statistics of the data traces

5.1.1. RWTH trace

Early 2000, we collected the access logs of the RWTH Aachen proxy server. The logs comprise the description of about 115 million HTTP and FTP requests made over a period of 54 days. After some preprocessing and filtering, about 17.3 million requests of interest remained. We studied the sizes of the objects requested by the clients. Fig. 3 shows the complementary distribution of the object sizes as
Fig. 3. RWTH trace: complementary log–log plot of document size distribution.

log–log plot. Some statistics for this trace are listed in Table 2. As can be observed from these statistics and the complementary distribution, the object-size distribution function decays much slower than a negative exponential distribution and is clearly heavy-tailed. We also observe a median much smaller than the mean.

5.1.2. NASA trace

The next trace we use for validation was first presented and evaluated in 1996 by Arlitt and Williamson [12]. It consists of about 3.1 million requests collected at the web server of the Kennedy Space Center. As in the RWTH trace, the size distribution of the requested objects in the NASA trace is clearly heavy-tailed, yielding a high coefficient of variation and a mean much larger than the median, as shown in Table 2.

5.2. Matching HTDs

5.2.1. RWTH trace

In Table 3, we show how well the two algorithms match the moments and median of the measurement data from the RWTH trace. Using the moment matching method (see Appendix B), the fitted Weibull

Table 2
Statistics for the object size (in bytes) for the RWTH and the NASA trace

<table>
<thead>
<tr>
<th>Trace</th>
<th>Min</th>
<th>Max</th>
<th>Mean</th>
<th>Median</th>
<th>CV²</th>
</tr>
</thead>
<tbody>
<tr>
<td>RWTH</td>
<td>118</td>
<td>$10^7$</td>
<td>6664</td>
<td>2638</td>
<td>6.12</td>
</tr>
<tr>
<td>NASA</td>
<td>3</td>
<td>$6.823 \times 10^9$</td>
<td>20744.9</td>
<td>4142</td>
<td>13.39</td>
</tr>
</tbody>
</table>
distribution exactly matches the first and second moment, but fails to fit the third moment and the median of the data set well. The three hyper-exponentials fitted with the FW-approach do not match the mean and squared coefficient of variation of the trace. For that reason, we did not compute their third moment and median. In contrast, both HEDs fitted with the new algorithm do fit these two statistics well. The relative error RE in percent (written in parenthesis), defined as $RE(x,y) = |(x - y)/y|\times 100\%$, is below 10% for all statistics.

5.2.2. NASA trace
The matching results for the NASA trace are shown in Table 4. They are comparable to those obtained for the RWTH trace. However, it seems that the higher degree of heavy-tailedness of the NASA trace results in larger relative errors for the matched HTDs.

5.3. Embedding HTDs in queuing models
In the second validation step, we used the fitted distribution for the RWTH trace as service-time distribution in an $M/G/1$ queue (modelling a proxy server). We studied mean and higher-order queuing measures for two different server loads ($\rho = 2/3$ resp. $\rho = 5/6$). The results are shown in Tables 5 and 6. For the measurement data and for the fitted Weibull distribution, the results were computed using a

### Table 3
RWTH trace: comparison of statistics of the measurement data, fitted HTDs and fitted HEDs

<table>
<thead>
<tr>
<th>Trace</th>
<th>$E[X]$</th>
<th>Weibull</th>
<th>Pareto</th>
<th>$FW:H_5$</th>
<th>$FW:H_{10}$</th>
<th>$FW:H_{20}$</th>
<th>EM $H_5$</th>
<th>EM $H_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6663.69</td>
<td>6663.69</td>
<td>6663.69</td>
<td>7025.25</td>
<td>5097.09</td>
<td>4977.77</td>
<td>6663.69</td>
<td>6663.69</td>
</tr>
<tr>
<td>CV²</td>
<td>6.12</td>
<td>6.12</td>
<td>6.12</td>
<td>1.949</td>
<td>4.20152</td>
<td>4.49</td>
<td>6.15 0.5%</td>
<td>6.22 1.8%</td>
</tr>
<tr>
<td>$E[X^3]$</td>
<td>$2.75 \times 10^{11}$</td>
<td>$4.04 \times 10^{11}$</td>
<td>n.d.</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>$2.91 \times 10^{11}$</td>
<td>$2.97 \times 10^{11}$</td>
</tr>
<tr>
<td>Median</td>
<td>2638</td>
<td>1322</td>
<td>4826</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>2044 9.7%</td>
<td>2640 &lt; 0.1%</td>
</tr>
</tbody>
</table>

### Table 4
NASA trace: comparison of statistics of the measurement data, fitted HTDs and fitted HEDs

<table>
<thead>
<tr>
<th>Trace</th>
<th>$E[X]$</th>
<th>Weibull</th>
<th>Pareto</th>
<th>$FW:H_5$</th>
<th>$FW:H_{10}$</th>
<th>$FW:H_{20}$</th>
<th>EM $H_5$</th>
<th>EM $H_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>20744.9</td>
<td>20744.9</td>
<td>20744.9</td>
<td>169.950</td>
<td>46829.5</td>
<td>12.475</td>
<td>20744.9</td>
<td>20744.9</td>
</tr>
<tr>
<td>CV²</td>
<td>13.39</td>
<td>13.39</td>
<td>13.39</td>
<td>1.006</td>
<td>1.2068</td>
<td>11.24</td>
<td>14.43 7.8%</td>
<td>14.43 7.8%</td>
</tr>
<tr>
<td>$E[X^3]$</td>
<td>$5.02 \times 10^{13}$</td>
<td>$6.77 \times 10^{11}$</td>
<td>n.d.</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>$6.62 \times 10^{11}$</td>
<td>$6.62 \times 10^{11}$</td>
</tr>
<tr>
<td>Median</td>
<td>4142</td>
<td>1750</td>
<td>15008</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>3717 10.3%</td>
<td>3717 10.3%</td>
</tr>
</tbody>
</table>

### Table 5
RWTH trace: queuing performance for load $\rho = 2/3$

<table>
<thead>
<tr>
<th>Trace</th>
<th>$E[N]_c$</th>
<th>$E[W]_c$</th>
<th>$c^2_1$</th>
<th>$c^2_2$</th>
<th>RE (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5.44</td>
<td>10.22</td>
<td>47.749</td>
<td>12.93</td>
<td>3.2%</td>
</tr>
<tr>
<td>Weibull</td>
<td>5.39</td>
<td>2.43</td>
<td>47.238</td>
<td>2.80</td>
<td>0.7%</td>
</tr>
<tr>
<td>$H_5$</td>
<td>5.34</td>
<td>10.91</td>
<td>47.657</td>
<td>13.82</td>
<td>6.9</td>
</tr>
<tr>
<td>$H_{10}$</td>
<td>5.49</td>
<td>10.93</td>
<td>48.208</td>
<td>13.81</td>
<td>6.8</td>
</tr>
</tbody>
</table>

### Table 6
NASA trace: queuing performance for load $\rho = 5/6$

<table>
<thead>
<tr>
<th>Trace</th>
<th>$E[N]_c$</th>
<th>$E[W]_c$</th>
<th>$c^2_1$</th>
<th>$c^2_2$</th>
<th>RE (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5.44</td>
<td>10.22</td>
<td>47.749</td>
<td>12.93</td>
<td>3.2%</td>
</tr>
<tr>
<td>Weibull</td>
<td>5.39</td>
<td>2.43</td>
<td>47.238</td>
<td>2.80</td>
<td>0.7%</td>
</tr>
<tr>
<td>$H_5$</td>
<td>5.34</td>
<td>10.91</td>
<td>47.657</td>
<td>13.82</td>
<td>6.9</td>
</tr>
<tr>
<td>$H_{10}$</td>
<td>5.49</td>
<td>10.93</td>
<td>48.208</td>
<td>13.81</td>
<td>6.8</td>
</tr>
</tbody>
</table>
Table 6
RWTH trace: queuing performance for load $\rho = \frac{5}{6}$

<table>
<thead>
<tr>
<th></th>
<th>$E[N_c]$</th>
<th>$c_n^2$</th>
<th>$E[W]$</th>
<th>$c_w^2$</th>
<th>RE (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trace</td>
<td>15.76</td>
<td>5.59</td>
<td>119,290</td>
<td>6.15</td>
<td>6.8%</td>
</tr>
<tr>
<td>Weibull</td>
<td>15.58</td>
<td>1.61</td>
<td>119,940</td>
<td>1.70</td>
<td>6.2%</td>
</tr>
<tr>
<td>$H_5$</td>
<td>15.73</td>
<td>5.58</td>
<td>119,144</td>
<td>6.13</td>
<td>0.3%</td>
</tr>
<tr>
<td>$H_{10}$</td>
<td>15.90</td>
<td>5.58</td>
<td>120,219</td>
<td>6.12</td>
<td>0.5%</td>
</tr>
</tbody>
</table>

Table 7
NASA trace: queuing performance for load $\rho = \frac{2}{3}$

<table>
<thead>
<tr>
<th></th>
<th>$E[N_c]$</th>
<th>$c_n^2$</th>
<th>$E[W]$</th>
<th>$c_w^2$</th>
<th>RE (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trace</td>
<td>10.32</td>
<td>2.61</td>
<td>300,353</td>
<td>2.81</td>
<td>0.2%</td>
</tr>
<tr>
<td>Weibull</td>
<td>10.22</td>
<td>3.19</td>
<td>297,247</td>
<td>3.48</td>
<td>3.1%</td>
</tr>
<tr>
<td>$H_5$</td>
<td>10.95</td>
<td>2.86</td>
<td>320,013</td>
<td>3.08</td>
<td>9.6%</td>
</tr>
<tr>
<td>$H_{10}$</td>
<td>10.95</td>
<td>2.86</td>
<td>320,013</td>
<td>3.08</td>
<td>9.6%</td>
</tr>
</tbody>
</table>

trace-driven and a stochastic discrete-event simulator, respectively, whereas the results for the fitted HEDs were obtained by numerical queueing analysis using our tool FiFiQueues [19], using a matrix-geometric solution for an M[HED]1 queue. Since the quality of the distributions provided by the FW-approach are limited by the quality of the fitted Weibull distribution, we refrained from using these here.

In both tables, we show the mean and the squared coefficient of variation of both queue length (including jobs in service) and waiting time. As can be seen, the first moment of queue length and waiting time is reasonably accurate for all distributions. For the squared coefficient of variation of the waiting time, denoted $c_w^2$, we also state the 95% confidence intervals relative to the mean (in parentheses, for the simulation results), as well as its relative errors RE in percent. We observe that the relative errors for the fitted HEDs are the smallest. Furthermore, a higher number of phases in the HED does not necessarily lead to a better fit in terms of queueing performance. It appears, once again, that the Weibull fit does not describe the statistical properties in the trace well.

Finally, we performed the same evaluations for the NASA trace. The results are shown in Tables 7 and 8, from which we can draw similar conclusions as from the RWTH trace, however, with slightly larger relative errors. Note that the distributions obtained by the EM-approach still provide better results than the Weibull distributions.

Table 8
NASA trace: queuing performance for load $\rho = \frac{5}{6}$

<table>
<thead>
<tr>
<th></th>
<th>$E[N_c]$</th>
<th>$c_n^2$</th>
<th>$E[W]$</th>
<th>$c_w^2$</th>
<th>RE (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trace</td>
<td>30.76</td>
<td>1.66</td>
<td>745,095</td>
<td>1.71</td>
<td>0.6%</td>
</tr>
<tr>
<td>Weibull</td>
<td>30.61</td>
<td>1.91</td>
<td>741,876</td>
<td>1.97</td>
<td>1.3%</td>
</tr>
<tr>
<td>$H_5$</td>
<td>32.97</td>
<td>1.78</td>
<td>800,018</td>
<td>1.83</td>
<td>7.0%</td>
</tr>
<tr>
<td>$H_{10}$</td>
<td>32.97</td>
<td>1.78</td>
<td>800,040</td>
<td>1.83</td>
<td>7.0%</td>
</tr>
</tbody>
</table>
6. Conclusions and final remarks

In this paper we have presented a new, direct way of fitting HEDs to measurement data that describes HTDs. In comparison to previous approaches, the new method may be computationally less attractive (it has a higher complexity), but its results are far more satisfying. The new method does not require any intermediate distribution function (form) to be chosen, and hence, delivers good approximations to the measurement data in cases where no closed-form HTD function is available or difficult to compute. The thus-obtained HEDs match the first and second moment as well as higher moments and shape characteristics like the median of the original distribution very well. We have shown that this improves the quality of queueing analysis results, especially for higher moments of queue measures.

At the same time, the proposed fitting procedure allows us to classify events (in the case study: objects-sizes), that is, a fitted HED tells us that with probability \( c_i \), an object “belongs to” class \( i \) of which the mean length is \( 1/\lambda_i \). Based on this observation, we have developed new scheduling and new caching algorithms for world-wide web servers, for which we have shown that these can improve the performance of such servers [16,17].

Acknowledgements

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Appendix A. The ML method

Let \( p(x|v) \) be a density function, parameterised by the vector \( v = (v_1, \ldots, v_M) \). Given measurement data \( x_1, \ldots, x_N \), we search a value for \( v \) (a so-called parameter estimator) so that the distribution with density \( p(x|v) \) “best” fits the distribution of the measurement data. In the ML method, the quality of the fitting is expressed by likelihood-function

\[
L(x_1, \ldots, x_N|v) = \prod_{i=1}^{N} p(x_i|v). \quad (A.1)
\]

The goal is to maximise \( L \) by choosing an appropriate value for \( v \). This can be achieved by solving the following set of equations:

\[
\frac{\partial L}{\partial v_1} = 0, \ldots, \frac{\partial L}{\partial v_M} = 0. \quad (A.2)
\]

Instead of (A.2), one often solves

\[
\frac{\partial \log L}{\partial \log v_1} = 0, \ldots, \frac{\partial \log L}{\partial \log v_M} = 0, \quad (A.3)
\]

since the log-function transforms the product into a sum.
We consider three cases:

- **The exponential distribution** with \( p(x|\lambda) = \lambda e^{-\lambda x} \) yields
  \[
  L(x_1, \ldots, x_N|\lambda) = \lambda^N e^{-\lambda \sum_{i=1}^{N} x_i},
  \]
  with “maximising” parameter estimator
  \[
  \lambda = \frac{N}{\sum_{i=1}^{N} x_i}.
  \]

- For the **Weibull distribution** with \( p(x|a,b) = \frac{b}{ab} x^{b-1} e^{-\left(x/a\right)^b} \), we find
  \[
  L(x_1, \ldots, x_N|a,b) = \left(\frac{b}{ab}\right)^b \prod_{i=1}^{N} x_i^{b-1} e^{-\left(\sum_{i=1}^{N} x_i/a\right)^b},
  \]
  which yields
  \[
  a = \left(\frac{\sum_{i=1}^{N} x_i^{b+1}}{N}\right)^{1/b} \quad \text{and} \quad \frac{1}{b} + \frac{1}{N} \sum_{i=1}^{N} \log x_i - \frac{\sum_{i=1}^{N} x_i \log(x_i)}{\sum_{i=1}^{N} x_i^b} = 0.
  \]
  This system of non-linear equations can be solved iteratively, e.g., with a Newton iteration method.

- For the **Pareto** distribution, we have \( p(x|a,k) = \frac{ak}{x^{a+1}} \) and \( F(x) = 1 - \left(x/k\right)^a \). The values for \( a \) and \( k \) can be computed via a least-squares fit to \( \log(1 - F(x)) = a \log(k) - a \log(x) \).

**Appendix B. Moment matching**

The moment matching method resembles the ML-approach: given measurement data \( x_1, \ldots, x_N \), we search the “best” value for the parameter vector \( \mathbf{v} = (v_1, \ldots, v_M) \) of the density function \( p(x|\mathbf{v}) \). To compute a value for \( \mathbf{v} \), we set the moments of the distribution to the moments of the measurement data and solve the system of equations resulting from them.

- For the **Weibull** density function \( p(x|a,b) = \frac{b}{ab} x^{b-1} e^{-\left(x/a\right)^b} \), the first and the second moment are given by
  \[
  E[X] = \frac{a}{b} \Gamma\left(1 + \frac{1}{b}\right), \quad E[X^2] = \frac{a^2}{b^2} \Gamma\left(1 + \frac{2}{b}\right).
  \]
  (B.1)
  The searched values for the parameters \( a \) and \( b \) can be computed by setting the first and second moment of the Weibull distribution to the first moment \( \bar{x} \) and the second moment \( \bar{x}^2 \) of the measurement data, respectively. We obtain
  \[
  \frac{E[X^2]}{E[X]^2} = \frac{2b \Gamma(2/b)}{\Gamma(1/b)^2} = \frac{\bar{x}^2}{\bar{x}^2}.
  \]
  (B.2)
From (B.2), we can compute the value of the parameter $b$ by using an iterative numerical method. Then, $a$ is given by

$$a = \frac{b\bar{x}}{F'(1/b)}. \quad \text{(B.3)}$$

- For the Pareto density function $p(x|a,k) = \left(\frac{ak}{a-1}\right)x^{-a}$, the first and second moment are given by

$$E[X] = \frac{ak}{a-1}, \quad E[X^2] = \frac{ak^2}{a-2}. \quad \text{(B.4)}$$

By setting the first and second moment of the Pareto distribution to the first and second moment of the measurement data, respectively, we obtain

$$\frac{E[X^2]}{E[X]} = \frac{(a-1)^2}{a(a-2)} = \frac{\bar{x}^2}{\bar{x}^2}. \quad \text{(B.5)}$$

From (B.5), we can compute the value of the parameter $a$ and then $k$ is given by

$$k = a - 1, a \bar{x}. \quad \text{(B.6)}$$

Appendix C. The EM method

The EM method is an extension of the ML method. Again measurement data $x_1, \ldots, x_N$ and a distribution to fit are given. In the EM method, the density function is the weighted sum of one or more “basic” densities $p(x|\theta_i)$, that is,

$$p(x|\alpha) = \sum_{i=1}^{I} c_i p(x|\theta_i),$$

where $\alpha = (c, \theta)$ is the parameter of the composed distribution with $c = (c_1, \ldots, c_I)$, and $\theta = (\theta_1, \ldots, \theta_I)$. For the mixture weights $c_i$, we assume $c_i \geq 0$ for $i = 1, \ldots, I$ and $\sum_{i=1}^{I} c_i = 1$.

As in the ML method, we search a parameter estimator $\hat{\alpha}$ that maximises the likelihood function $L(x_1, \ldots, x_N|\alpha)$. Alternatively, we can define the optimal estimator as the estimator $\hat{\alpha}$ that maximises the difference $D(\alpha, \hat{\alpha}) = log L(x_1, \ldots, x_N|\hat{\alpha}) - log L(x_1, \ldots, x_N|\alpha)$ for any other estimator $\alpha$. Developing $D$ we obtain

$$D(\alpha, \hat{\alpha}) = \sum_{n=1}^{N} \log p(x_n|\hat{\alpha}) - \sum_{n=1}^{N} \log p(x_n|\alpha) = \sum_{n=1}^{N} \log \left(\frac{p(x_n|\hat{\alpha})}{p(x_n|\alpha)}\right). \quad \text{(C.1)}$$

$$D(\alpha, \hat{\alpha}) = \sum_{n=1}^{N} \sum_{y \in Y} \delta(y|x_n, \alpha) \log \left(\frac{p(y|x_n, \hat{\alpha})}{p(y|x_n, \alpha)}\right). \quad \text{(C.2)}$$
In the last equation, we have introduced a so-called hidden variable $y$ with distribution $\delta(y|x_n, \alpha)$. Using $y$ we can define a two-dimensional density function $\delta(x_n, y|\alpha) = p(x_n|\alpha)\delta(y|x_n, \alpha)$ and we obtain (by the use of the law of conditional probabilities):

$$D(\alpha, \hat{\alpha}) = \sum_{n=1}^{N} \sum_{y} \delta(y|x_n, \alpha) \log \frac{\delta(x_n, y|\hat{\alpha})}{\delta(x_n, y|\alpha)} \tag{C.3}$$

$$D(\alpha, \hat{\alpha}) = \sum_{n=1}^{N} \sum_{y} \delta(y|x_n, \alpha) \log \frac{\delta(x_n, y|\hat{\alpha})}{\delta(x_n, y|\alpha)} + \sum_{n=1}^{N} \sum_{y} \delta(y|x_n, \alpha) \log \frac{\delta(y|x_n, \alpha)}{\delta(y|x_n, \hat{\alpha})} \tag{C.4}$$

It can be shown that the second additive term $\sum_{n=1}^{N} \sum_{y} \delta(y|x_n, \alpha) \log (\delta(y|x_n, \alpha)/\delta(y|x_n, \hat{\alpha})) \geq 0$. The proof of this statement is simple: since it holds $\log(t) \leq t - 1$, for $t > 0$, we have

$$\sum_{y} p(y) \log \frac{p(y)}{q(y)} = -\sum_{y} p(y) \log \frac{q(y)}{p(y)} \geq -\sum_{y} p(y) \left( \frac{q(y)}{p(y)} - 1 \right) = -\sum_{y} q(y) + \sum_{y} p(y) = -1 + 1 = 0.$$  

From this it follows that

$$D(\alpha, \hat{\alpha}) \geq \sum_{n=1}^{N} \sum_{y} \delta(y|x_n, \alpha) \log \frac{\delta(x_n, y|\hat{\alpha})}{\delta(x_n, y|\alpha)}$$

or, by defining $Q(\alpha, \hat{\alpha}) = \sum_{n=1}^{N} \sum_{y} \delta(y|x_n, \alpha) \log (\delta(y|x_n, \alpha)/\delta(y|x_n, \hat{\alpha})): \quad D(\alpha, \hat{\alpha}) \geq Q(\alpha, \hat{\alpha}) - Q(\alpha, \alpha). \tag{C.5}$$

The EM-algorithm uses the following iteration scheme to find the optimal estimator $\alpha$:

1. Choose an initial estimator $\alpha$.
2. Calculate a better estimator $\alpha' = \text{argmax}_{\alpha} Q(\alpha, \hat{\alpha})$.
3. Continue iteration with $\alpha := \alpha'$ if $|\alpha - \alpha'| > \epsilon$.

To compute $\alpha'$ in step 2, we first have to define the hidden variable $y$ and its density function. We choose $y \in \{1, \ldots, I\}$ with density function $\delta(y|x_n, \alpha) = \frac{c_i p(x_n|\theta_i)}{p(x_n|\alpha)}$.

Now, $\alpha' = (\alpha_i', \theta_i')$ can be computed by solving the non-linear equation:

$$\frac{\partial Q}{\partial \theta_i} = \sum_{n=1}^{N} \delta(i|x_n, \theta_i) \frac{\partial}{\partial \theta_i} \log (c_i p(x_n|\theta_i')) = 0, \quad i = 1, \ldots, I. \tag{C.6}$$

Using a Lagrange multiplier, one obtains [18]

$$c_i' = \frac{1}{N} \sum_{n=1}^{N} \delta(i|x_n, \theta_i') = \frac{1}{N} \sum_{n=1}^{N} \frac{c_i p(x_n|\theta_i)}{p(x_n|\alpha)}, \quad i = 1, \ldots, I. \tag{C.7}$$
Table 9 Basic distributions and their iteration formulae for the EM-algorithm

| Distribution | Parameters | $p(x_n|\theta_i)$ | Iterations |
|--------------|------------|-------------------|------------|
| Exponential  | $(\lambda_i)$ | $e^{-\lambda_1 x_n}$ | $\lambda_i' = \sum_{n=1}^{N} p(i|x_n,\theta_i)\lambda_k / \sum_{n=1}^{N} p(i|x_n,\theta_i)$ |
| Gauss        | $(\mu_i,\sigma_i)$ | $\frac{1}{\sqrt{2\pi\sigma^2_i}} e^{-\frac{(x_n-\mu_i)^2}{2\sigma^2_i}}$ | $\mu_i' = \sum_{n=1}^{N} p(i|x_n,\theta_i) x_n / \sum_{n=1}^{N} p(i|x_n,\theta_i)$, $\sigma_i'^2 = \sum_{n=1}^{N} p(i|x_n,\theta_i) (x_n - \mu_i')^2 / \sum_{n=1}^{N} p(i|x_n,\theta_i)$ |
| Lognormal    | $(\mu_i,\sigma_i)$ | $\frac{1}{\sqrt{2\pi\sigma^2_i}} e^{-\frac{\ln x_n - \mu_i^2}{2\sigma^2_i}}$ | $\mu_i' = \sum_{n=1}^{N} p(i|x_n,\theta_i) \ln x_n / \sum_{n=1}^{N} p(i|x_n,\theta_i)$, $\sigma_i'^2 = \sum_{n=1}^{N} p(i|x_n,\theta_i) (\ln x_n - \mu_i')^2 / \sum_{n=1}^{N} p(i|x_n,\theta_i)$ |

Table 9 gives an overview over some interesting distributions and their explicit solutions for $\theta'$. Note that we did only use the result for the exponential distribution.

References


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